To the Vector Go the Spoils: Motivating an Introduction to Vectors Using Interactive Computer Graphics

Bruce Cohen* and David Sklar†

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*Lowell High School
San Francisco, CA 94132
bic@cgl.ucsf.edu
http://www.cgl.ucsf.edu/home/bic

†SOLA Optical USA
dsklar46@yahoo.com

Outline

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1. The Unit Problem — A Virtual Trackball
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4. Vector Product
5. Using Vectors — Exploring a Parallelepiped
6. A Rotation Function Using Vectors
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Vector Notation

a is a scalar.

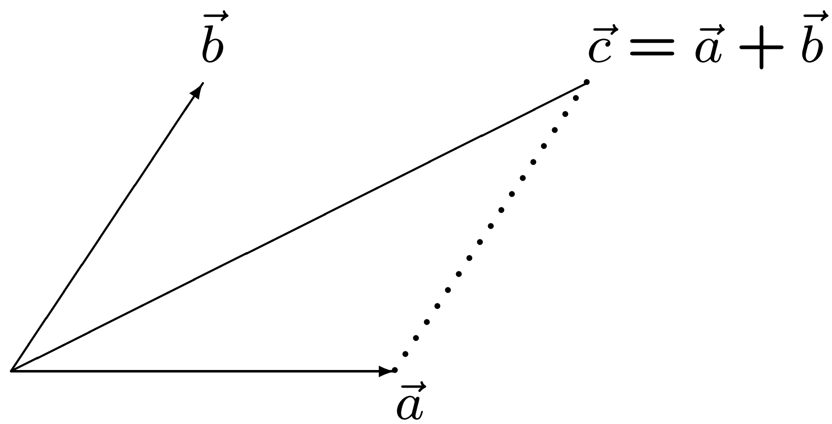
$\vec{v}$ is vector.

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$|\vec{v}|$ is the magnitude of $\vec{v}$.

$\hat{v}$ is a unit vector.

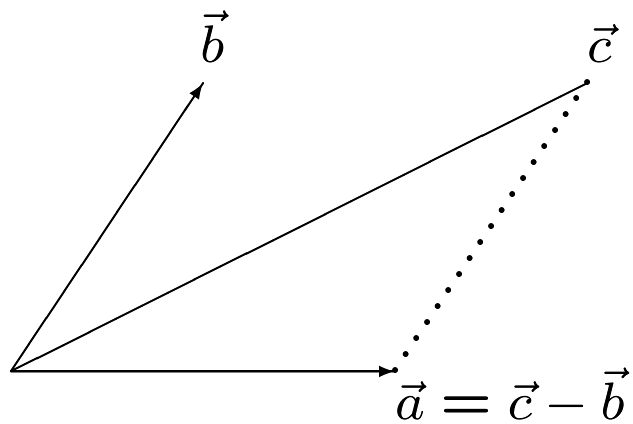
Vector Addition



$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

Vector Subtraction



$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} c_1 - b_1 \\ c_2 - b_2 \\ c_3 - b_3 \end{pmatrix}$$

Product of a Scalar and a Vector

$$a\vec{c} = a \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \begin{pmatrix} ac_1 \\ ac_2 \\ ac_3 \end{pmatrix}$$

Magnitude of a Vector

Look at Kinemage 2

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$|\vec{c}| = \sqrt{c_1^2 + c_2^2 + c_3^2}$$

Magnitude of the Product of a Scalar and a Vector

$$|a\vec{c}| = \sqrt{(ac_1)^2 + (ac_2)^2 + (ac_3)^2}$$

$$= \sqrt{a^2(c_1^2 + c_2^2 + c_3^2)}$$

$$= a\sqrt{c_1^2 + c_2^2 + c_3^2}$$

$$= a|\vec{c}|$$

Unit Vectors

Assume some vector $\vec{v}$. Let $\widehat{v} = \dfrac{1}{|\vec{v}|}\vec{v}$.

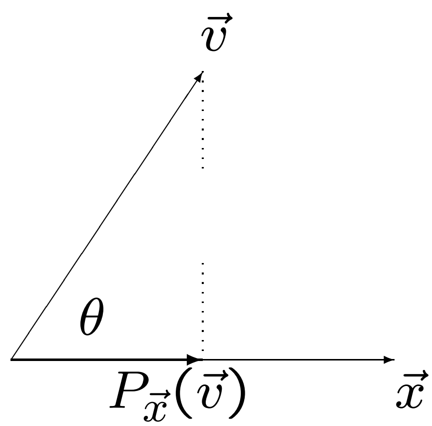
We can write this as $\widehat{v} = \dfrac{\vec{v}}{|\vec{v}|}$.

Parameterized Functions

- $F_{a,b}(x)$ is a parameterized function with parameters a and b and variable x.

- Consider $L_{M,B}(x) = Mx + B$.

Orthogonal Projection of a Vector



$$P_{\vec{x}}(\vec{v}) = \frac{|\vec{v}| \cos\theta}{|\vec{x}|}\vec{x}$$

Using a unit vector:

$$P_{\widehat{x}}(\vec{v}) = |\vec{v}| \cos\theta \, \widehat{x}$$

Examples of Projections

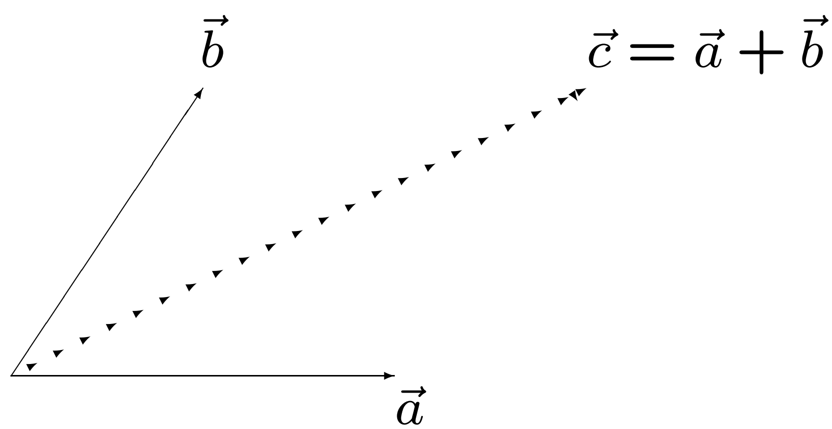
$$P_{\hat{i}}(\vec{v}) = |\vec{v}| \cos\theta \, \hat{i}$$

$$= v_1 \hat{i}$$

$$P_{\hat{j}}(\vec{v}) = v_2 \hat{j}$$

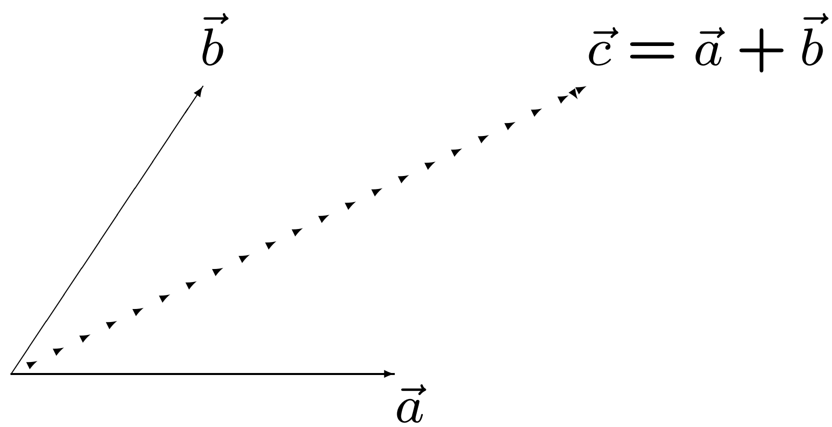
$$P_{\hat{k}}(\vec{v}) = v_3 \hat{k}$$

Magnitude of a Vector Sum



Given: $\vec{c} = \vec{a} + \vec{b}$ find $|\vec{c}|$

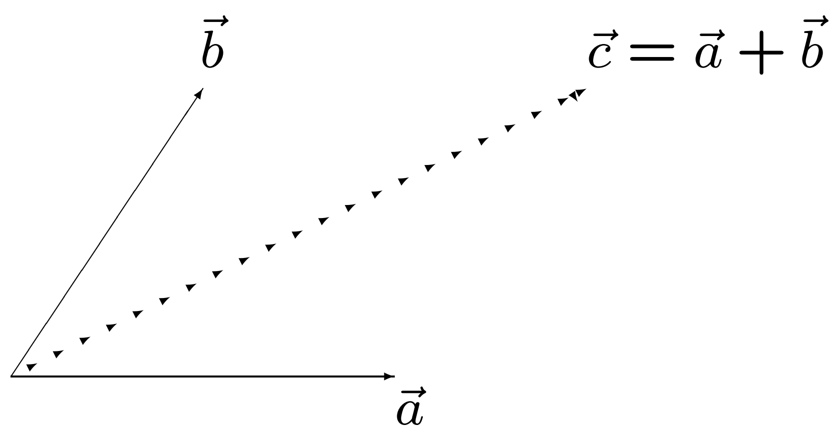
Magnitude of a Vector Sum 1A



$$\vec{c} = \vec{a} + \vec{b}$$

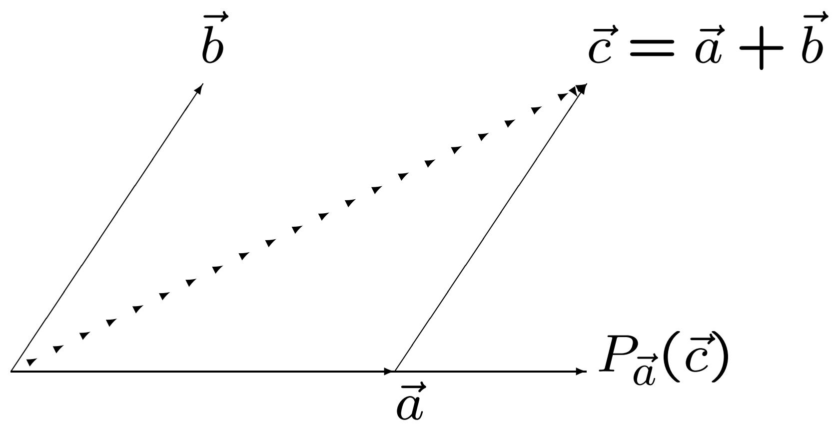
$$\begin{aligned}
|\vec{c}|^2 &= (a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2 \\
&= (a_1^2 + 2a_1 b_1 + b_1^2) + (a_2^2 + 2a_2 b_2 + b_2^2) + \\
&\qquad (a_3^2 + 2a_3 b_3 + b_3^2) \\
&= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) + \\
&\qquad 2a_1 b_1 + 2a_2 b_2 + 2a_3 b_3 \\
&= |\vec{a}|^2 + |\vec{b}|^2 + 2(a_1 b_1 + a_2 b_2 + a_3 b_3)
\end{aligned}$$

Magnitude of a Vector Sum 2A



Given: $\vec{c} = \vec{a} + \vec{b}$ find $|\vec{c}|$

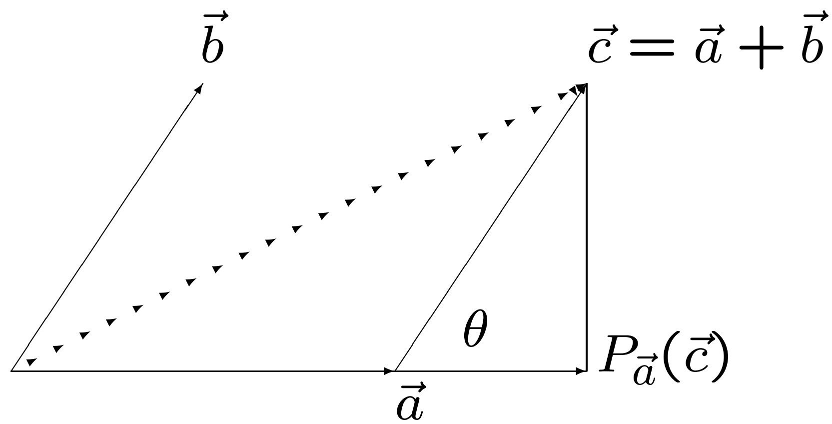
Magnitude of a Vector Sum 2B



Given: $\vec{c} = \vec{a} + \vec{b}$ find $|\vec{c}|$

Reposition $\vec{b}$
Look at $P_{\vec{a}}(\vec{c})$

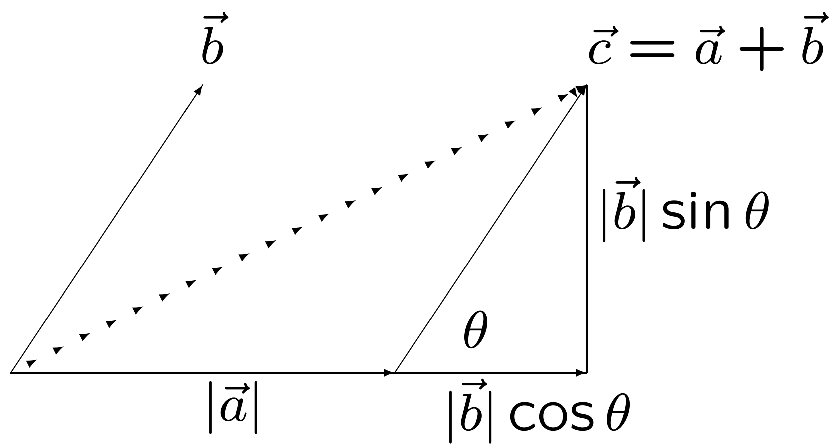
Magnitude of a Vector Sum 2C



Given: $\vec{c} = \vec{a} + \vec{b}$ find $|\vec{c}|$
Reposition $\vec{b}$
Look at $P_{\vec{a}}(\vec{c})$

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

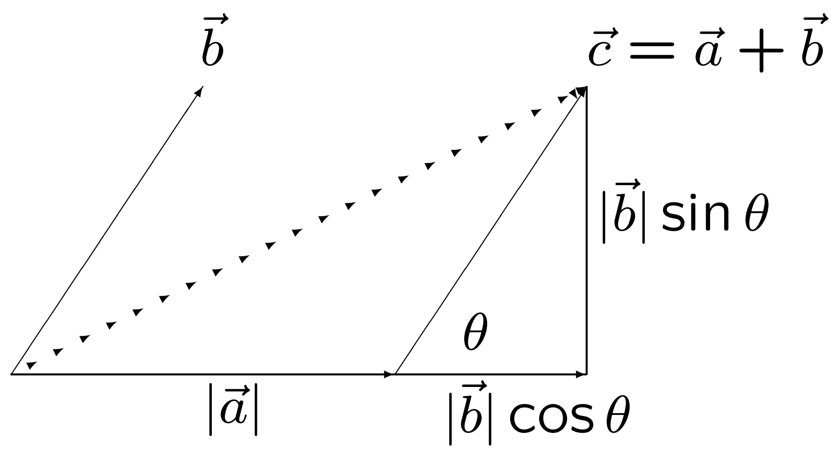
Magnitude of a Vector Sum 2D



Given: $\vec{c} = \vec{a} + \vec{b}$ find $|\vec{c}|$
Reposition $\vec{b}$
Look at $P_{\vec{a}}(\vec{c})$
Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Use θ to get lengths for some line segments.

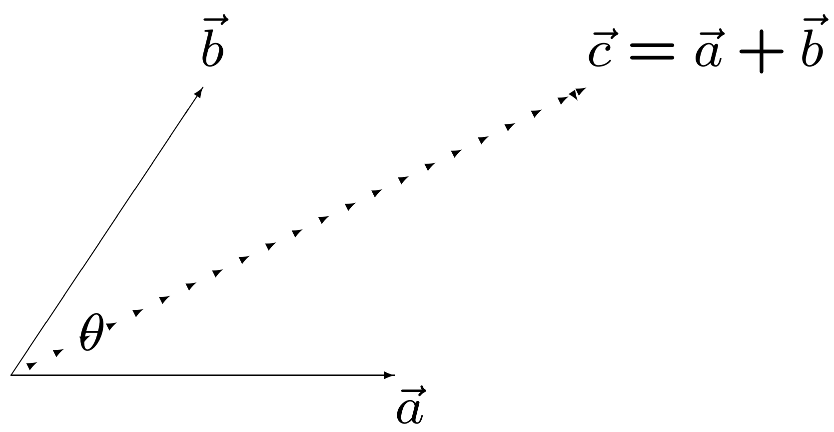
Magnitude of a Vector Sum 2D



Given: $\vec{c} = \vec{a} + \vec{b}$ find $|\vec{c}|$

$$
\begin{aligned}
|\vec{c}|^2 &= (|\vec{a}| + |\vec{b}| \cos\theta)^2 + (|\vec{b}| \sin\theta)^2 \\
&= |\vec{a}|^2 + 2|\vec{a}||\vec{b}| \cos\theta + |\vec{b}|^2 \cos^2\theta + |\vec{b}|^2 \sin^2\theta \\
&= |\vec{a}|^2 + 2|\vec{a}||\vec{b}| \cos\theta + |\vec{b}|^2 (\cos^2\theta + \sin^2\theta) \\
&= |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}| \cos\theta
\end{aligned}
$$

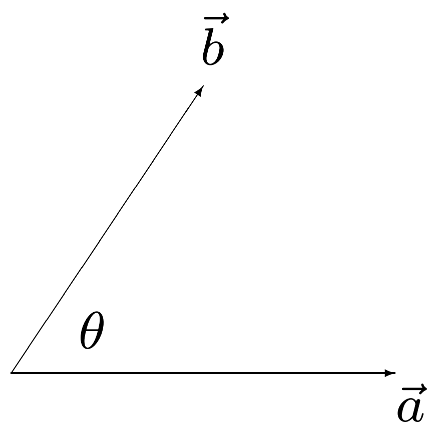
Magnitude of a Vector Sum Summary



$$\vec{c} = \vec{a} + \vec{b}$$

$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2(a_1 b_1 + a_2 b_2 + a_3 b_3)$$
$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}| \cos\theta$$
$$\therefore (a_1 b_1 + a_2 b_2 + a_3 b_3) = |\vec{a}||\vec{b}| \cos\theta$$

Dot Product



We can define the dot product as either:

$$\vec{a} \cdot \vec{b} = (a_1 b_1 + a_2 b_2 + a_3 b_3)$$

or

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$$

with the alternate as a theorem.

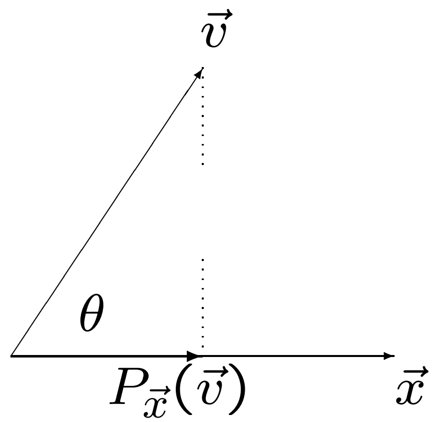
It is left as an exercise to show: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

Magnitude of a Vector Revised

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\begin{aligned} |\vec{c}| &= \sqrt{c_1^2 + c_2^2 + c_3^2} \\ \vec{c} \cdot \vec{c} &= c_1^2 + c_2^2 + c_3^2 \\ \therefore \vec{c} \cdot \vec{c} &= |\vec{c}|^2 \end{aligned}$$

Projection of a Vector using a Dot Product



Recall: $P_{\vec{x}}(\vec{v}) = \frac{|\vec{v}|\cos\theta}{|\vec{x}|}\vec{x}$

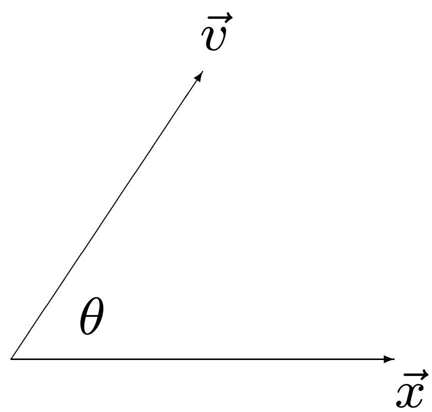
Since $\vec{x}\cdot\vec{v} = |\vec{x}||\vec{v}|\cos\theta$, we have $\cos\theta = \frac{\vec{x}\cdot\vec{v}}{|\vec{x}||\vec{v}|}$.

So: $P_{\vec{x}}(\vec{v}) = \frac{\vec{x}\cdot\vec{v}}{\vec{x}\cdot\vec{x}}\vec{x}$

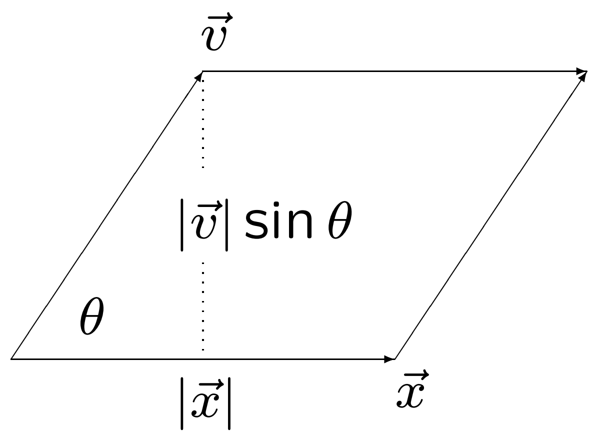
For projecting on to a unit vector $\hat{u}$,

$$P_{\hat{u}}(\vec{v}) = (\hat{u}\cdot\vec{v})\,\hat{u}$$

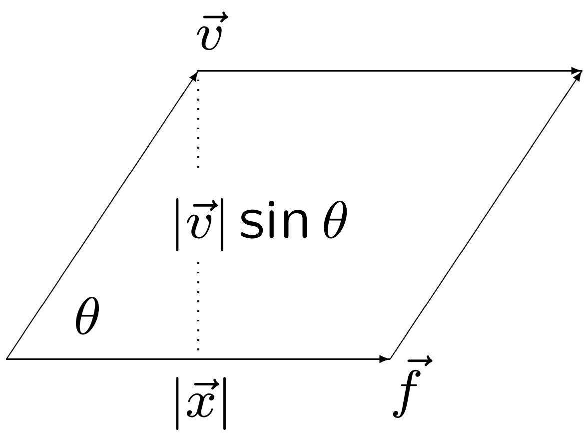
What about $|\vec{x}||\vec{v}|\sin\theta$?



What about $|\vec{x}||\vec{v}|\sin\theta$?



What about $|\vec{x}||\vec{v}|\sin\theta$ (Area)?



$$
\begin{aligned}
\text{Area}^2 &= \vec{x}\cdot\vec{x}\,\vec{v}\cdot\vec{v}\,\sin^2\theta \\
&= \vec{x}\cdot\vec{x}\,\vec{v}\cdot\vec{v}\,(1-\cos^2\theta) \\
&= \vec{x}\cdot\vec{x}\,\vec{v}\cdot\vec{v} - \vec{x}\cdot\vec{x}\,\vec{v}\cdot\vec{v}\,\cos^2\theta \\
&= \vec{x}\cdot\vec{x}\,\vec{v}\cdot\vec{v} - (\vec{x}\cdot\vec{v})^2
\end{aligned}
$$

What about $|\vec{x}||\vec{v}| \sin\theta$?
Connect the Dot (Products)

$$
\begin{aligned}
\text{Area}^2 =& \vec{x}\cdot\vec{x}\,\vec{v}\cdot\vec{v}\,\sin^2\theta \\
=& \vec{x}\cdot\vec{x}\,\vec{v}\cdot\vec{v} - (\vec{x}\cdot\vec{v})^2 \\
=& (x_1^2 + x_2^2 + x_3^2)(v_1^2 + v_2^2 + v_3^2) - \\
& \qquad (x_1v_1 + x_2v_2 + x_3v_3)^2 \\
=& x_1^2v_1^2 + x_1^2v_2^2 + x_1^2v_3^2 + \\
& x_2^2v_1^2 + x_2^2v_2^2 + x_2^2v_3^2 + \\
& x_3^2v_1^2 + x_3^2v_2^2 + x_3^2v_3^2 \\
& \qquad - (x_1v_1)^2 - x_1v_1x_2v_2 - x_1v_1x_3v_3 \\
& \qquad - (x_2v_2)^2 - x_2v_2x_1v_1 - x_2v_2x_3v_3 \\
& \qquad - (x_3v_3)^2 - x_3v_3x_1v_1 - x_3v_3x_2v_2
\end{aligned}
$$

What about $|\vec{x}||\vec{v}|\sin\theta$?
More Connect the Dot (Products)

$$
\begin{aligned}
\text{Area}^2 =& \vec{x}\cdot\vec{x}\,\vec{v}\cdot\vec{v}\,\sin^2\theta \\
=& x_1^2v_1^2 + x_1^2v_2^2 + x_1^2v_3^2 + \\
& x_2^2v_1^2 + x_2^2v_2^2 + x_2^2v_3^2 + \\
& x_3^2v_1^2 + x_3^2v_2^2 + x_3^2v_3^2 \\
& \qquad - (x_1v_1)^2 - x_1v_1x_2v_2 - x_1v_1x_3v_3 \\
& \qquad - (x_2v_2)^2 - x_2v_2x_1v_1 - x_2v_2x_3v_3 \\
& \qquad - (x_3v_3)^2 - x_3v_3x_1v_1 - x_3v_3x_2v_2
\end{aligned}
$$

$$
\begin{aligned}
\text{Area}^2 =& x_1^2v_2^2 + x_1^2v_3^2 - x_1v_1x_2v_2 - x_1v_1x_3v_3 + \\
& x_2^2v_1^2 + x_2^2v_3^2 - x_2v_2x_1v_1 - x_2v_2x_3v_3 + \\
& x_3^2v_1^2 + x_3^2v_2^2 - x_3v_3x_1v_1 - x_3v_3x_2v_2
\end{aligned}
$$

What about $|\vec{x}||\vec{v}|\sin\theta$?
Even More Connect the Dot (Products)

$$\begin{aligned}
\text{Area}^2 =& \vec{x}\cdot\vec{x}\,\vec{v}\cdot\vec{v}\,\sin^2\theta \\
=& x_1^2v_2^2 + x_1^2v_3^2 - x_1v_1x_2v_2 - x_1v_1x_3v_3 + \\
& x_2^2v_1^2 + x_2^2v_3^2 - x_2v_2x_1v_1 - x_2v_2x_3v_3 + \\
& x_3^2v_1^2 + x_3^2v_2^2 - x_3v_3x_1v_1 - x_3v_3x_2v_2 \\
=& x_1^2v_2^2 - 2x_1v_2x_2v_1 + x_2^2v_1^2 + \\
& x_1^2v_3^2 - 2x_1v_3x_3v_1 + x_3^2v_1^2 + \\
& x_2^2v_3^2 - 2x_2v_3x_3v_2 + x_3^2v_2^2 \\
=& (x_1v_2 - x_2v_1)^2 + \\
& (x_1v_3 - x_3v_1)^2 + \\
& (x_2v_3 - x_3v_2)^2
\end{aligned}$$

Vectors with a Magnitude of $|\vec{x}||\vec{v}|\sin\theta$

$$\begin{aligned}\text{Area}^2 =& \vec{x}\cdot\vec{x}\,\vec{v}\cdot\vec{v}\,\sin^2\theta \\ =& (x_1v_2 - x_2v_1)^2 + \\ & (x_1v_3 - x_3v_1)^2 + \\ & (x_2v_3 - x_3v_2)^2\end{aligned}$$

$$\begin{pmatrix} x_1v_2 - x_2v_1 \\ x_1v_3 - x_3v_1 \\ x_2v_3 - x_3v_2 \end{pmatrix} \cdot \begin{pmatrix} x_1v_2 - x_2v_1 \\ x_1v_3 - x_3v_1 \\ x_2v_3 - x_3v_2 \end{pmatrix} = \vec{x}\cdot\vec{x}\,\vec{v}\cdot\vec{v}\,\sin^2\theta$$

$$\text{Let } \vec{w} = \begin{pmatrix} x_1v_2 - x_2v_1 \\ x_1v_3 - x_3v_1 \\ x_2v_3 - x_3v_2 \end{pmatrix} \text{ then } |\vec{w}| = |\vec{x}||\vec{v}|\sin\theta$$

Vectors with a Magnitude of $|\vec{x}||\vec{v}|\sin\theta$

Let $\vec{w} = \begin{pmatrix} x_1v_2 - x_2v_1 \\ x_1v_3 - x_3v_1 \\ x_2v_3 - x_3v_2 \end{pmatrix}$ then $|\vec{w}| = |\vec{x}||\vec{v}|\sin\theta$

How many "equivalent" vectors are possible?

Vectors with a Magnitude of $|\vec{x}||\vec{v}|\sin\theta$

$$\text{Let } \vec{w} = \begin{pmatrix} x_2v_3 - x_3v_2 \\ x_3v_1 - x_1v_3 \\ x_1v_2 - x_2v_1 \end{pmatrix} \text{ then } |\vec{w}| = |\vec{x}||\vec{v}|\sin\theta$$

- How many vectors “equivalent” vectors are possible?

- We want $\vec{w} \cdot \vec{x} = 0$ and $\vec{w} \cdot \vec{v} = 0$.

- Right hand rule.

Cross Product:

$$\vec{x} \times \vec{v} \equiv \begin{pmatrix} x_2 v_3 - x_3 v_2 \\ x_3 v_1 - x_1 v_3 \\ x_1 v_2 - x_2 v_1 \end{pmatrix}$$

- Vector with a Magnitude of $|\vec{x}||\vec{v}| \sin\theta$.

- Direction orthogonal to both $\vec{x}$ and $\vec{v}$ obeying right hand rule.

Look at Java crossproduct applet.

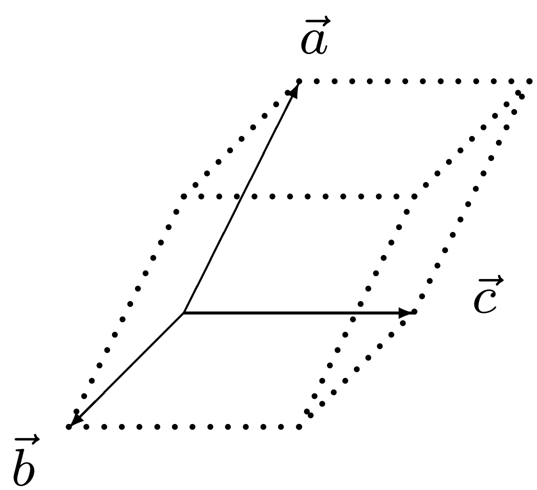
Cross Product Properties

$$\vec{x} \times \vec{v} \equiv \begin{pmatrix} x_2 v_3 - x_3 v_2 \\ x_1 v_3 - x_3 v_1 \\ x_1 v_2 - x_2 v_1 \end{pmatrix}$$

- $\hat{i} \times \hat{j} = \hat{k} \quad \hat{j} \times \hat{k} = \hat{i} \quad \hat{k} \times \hat{i} = \hat{j}$
- Cross product is *not* commutative: $\vec{x} \times \vec{v} \neq \vec{v} \times \vec{x}$ (In fact $\vec{x} \times \vec{v} \equiv -\vec{v} \times \vec{x}$)
- Cross product is (generally) *not* associative: $\vec{x} \times (\vec{v} \times \vec{z}) \neq (\vec{x} \times \vec{v}) \times \vec{z}$
- Cross product distributes over vector addition: $\vec{x} \times (\vec{v} + \vec{w}) \equiv \vec{x} \times \vec{v} + \vec{x} \times \vec{w}$

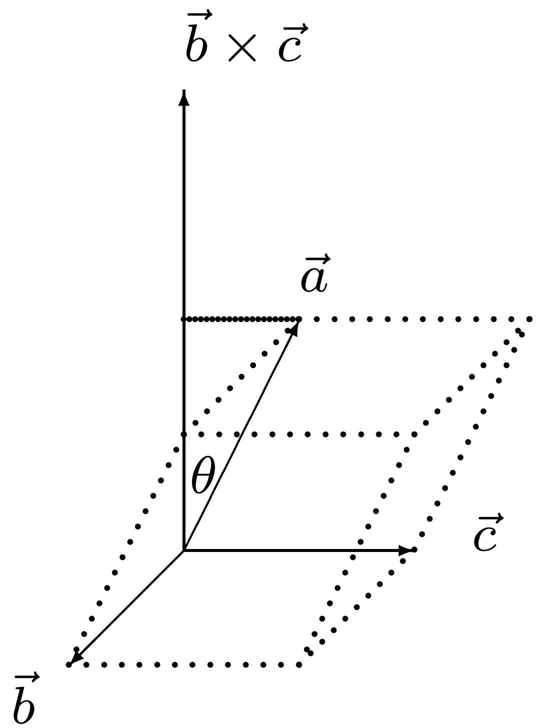
Take a Look at a Parallelepiped

Two vectors define a parallelogram. Three vectors define a parallelepiped.



Look at kinemage 3.

Volume of a Parallelepiped



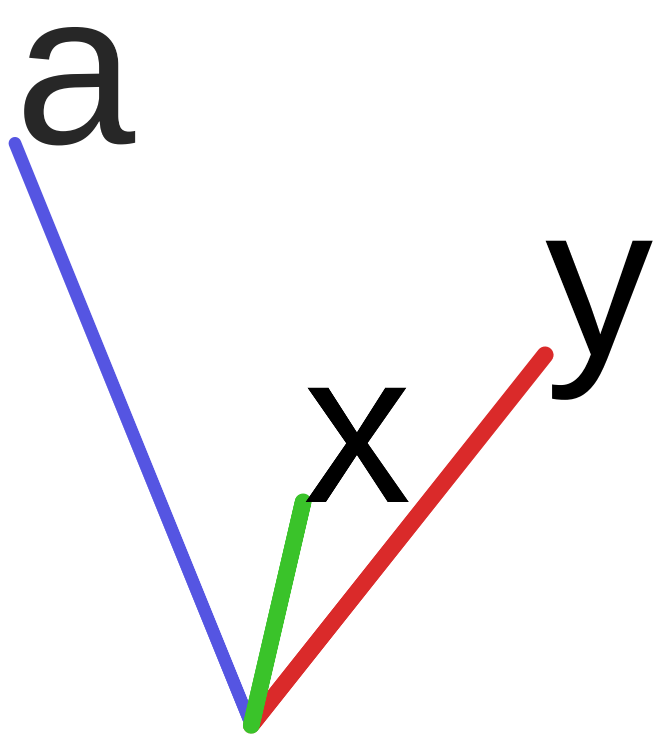
$|\vec{b} \times \vec{c}|$ is the area of the base parallelogram.
$|\vec{a}| \cos \theta$ is the height of the parallelepiped.
$|\vec{b} \times \vec{c}||\vec{a}| \cos \theta$ is the volume the parallelepiped.
$|\vec{b} \times \vec{c}||\vec{a}| \cos \theta = |\vec{a}||\vec{b} \times \vec{c}| \cos \theta = \vec{a} \cdot (\vec{b} \times \vec{c})$

Parameterized Functions

- $F_{a,b}(x)$ is a parameterized function with parameters a and b and variable x.

- Consider $L_{M,B}(x) = Mx + B$.

Rotating a Vector Around an Arbitrary Axis



Given a unit vector, $\hat{a}$, and an angle, θ, develop a function, $R_{\hat{a},\theta}(\vec{f})$ that will produce $\vec{y}$, the rotation of $\vec{f}$ around $\hat{a}$ by θ.

$\vec{x_\parallel}$ and $\vec{x_\perp}$

$$\vec{x_\parallel} = P_{\widehat{a}}(\vec{x})$$
$$\vec{x_\parallel} = (\vec{x} \cdot \widehat{a})\widehat{a}$$
$$\vec{x_\perp} = \vec{x} - \vec{x_\parallel}$$
$$\vec{x_\perp} = \vec{x} - (\vec{x} \cdot \widehat{a})\widehat{a}$$

$\vec{y_{\|}}$ and $\vec{y_{\perp}}$

$$\vec{y} = \vec{y_{\|}} + \vec{y_{\perp}}$$
$$\vec{y_{\|}} = \vec{x_{\|}}$$
$$\vec{y_{\|}} = (\vec{x} \cdot \hat{a})\hat{a}$$

We need $\vec{y_{\perp}}$ in terms of $\hat{a}, \vec{x}$ and θ.

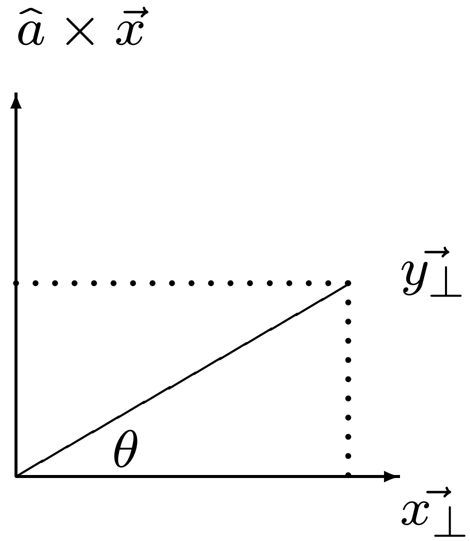
A Look at $\hat{a} \times \vec{x}$

$$
\begin{aligned}
\hat{a} \times \vec{x} &= \hat{a} \times (\vec{x_\parallel} + \vec{x_\perp}) \\
&= \hat{a} \times \vec{x_\parallel} + \hat{a} \times \vec{x_\perp} \\
&= \vec{0} + \hat{a} \times \vec{x_\perp} \\
&= \hat{a} \times \vec{x_\perp}
\end{aligned}
$$

$$
\begin{aligned}
|\hat{a} \times \vec{x}| &= |\hat{a} \times \vec{x_\perp}| \\
&= |\hat{a}||\vec{x_\perp}| \sin \frac{\pi}{2} \\
&= |\vec{x_\perp}|
\end{aligned}
$$

$$
|\vec{x_\perp}| = |\vec{y_\perp}|
$$

The $\vec{x_\perp}\vec{y_\perp}$ Plane



$$\hat{a} \times \vec{x} = \hat{a} \times \vec{x_\perp}$$
$$|\hat{a} \times \vec{x}| = |\vec{x_\perp}| = |\vec{y_\perp}|$$

$$\vec{y_\perp} = \cos\theta\, \vec{x_\perp} + \sin\theta\, \hat{a} \times \vec{x}$$

$R_{\hat{a},\theta}(\vec{x})$ at last

$$
\begin{aligned}
\vec{y} &= \vec{y_{\|}} + \vec{y_{\perp}} \\
\vec{y_{\|}} &= (\vec{x} \cdot \hat{a})\hat{a} \\
\vec{y_{\perp}} &= \cos\theta\, \vec{x_{\perp}} + \sin\theta\, \hat{a} \times \vec{x} \\
\vec{x_{\perp}} &= \vec{x} - (\vec{x} \cdot \hat{a})\hat{a}
\end{aligned}
$$

$$
\begin{aligned}
\vec{y} &= (\vec{x} \cdot \hat{a})\hat{a} + \cos\theta\,(\vec{x} - (\vec{x} \cdot \hat{a})\hat{a}) + \sin\theta\, \hat{a} \times \vec{x} \\
&= (1 - \cos\theta)(\vec{x} \cdot \hat{a})\hat{a} + \cos\theta\, \vec{x} + \sin\theta\, \hat{a} \times \vec{x}
\end{aligned}
$$

How about $M_{\hat{f},\hat{t}}(\vec{x})$?

$$
\begin{aligned}
R_{\hat{a},\theta}(\vec{x}) =& (1-\cos\theta)(\vec{x}\cdot\hat{a})\hat{a} + \cos\theta\,\vec{x} + \sin\theta\,\hat{a}\times\vec{x} \\
\hat{a} =& \frac{\hat{f}\times\hat{t}}{|\hat{f}\times\hat{t}|} \\
\cos\theta =& \hat{f}\cdot\hat{t} \\
\sin\theta =& \sqrt{1-\cos^2\theta} = \sqrt{1-(\hat{f}\cdot\hat{t}\,)^2}
\end{aligned}
$$

$$
\begin{aligned}
M_{\hat{f},\hat{t}}(\vec{x}) =& (1-\hat{f}\cdot\hat{t}\,)\left(\vec{x}\cdot\frac{\hat{f}\times\hat{t}}{|\hat{f}\times\hat{t}|}\right)\frac{\hat{f}\times\hat{t}}{|\hat{f}\times\hat{t}|} + \\
& (\hat{f}\cdot\hat{t}\,)\,\vec{x} + \sqrt{1-(\hat{f}\cdot\hat{t}\,)^2}\,\frac{\hat{f}\times\hat{t}}{|\hat{f}\times\hat{t}|}\times\vec{x}
\end{aligned}
$$

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