Polya's Orchard: Minkowski's Theorem

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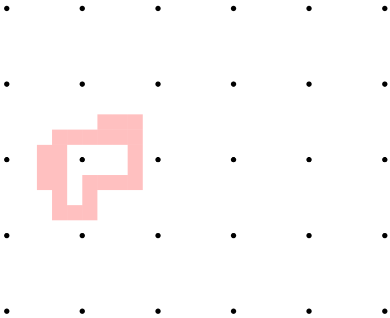
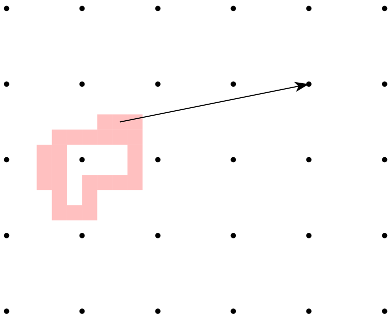
A Region and a Lattice

This region (call it $\mathcal{R}$) does not **contain** any *lattice point*.



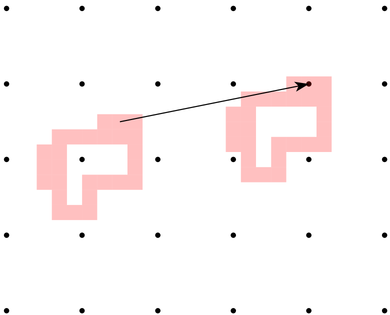
Translation

By selecting any point in $\mathcal{R}$ and a lattice point,



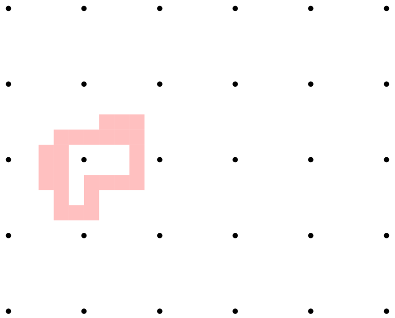
Translation

By selecting any point in $\mathcal{R}$ and a lattice point, we can make a translation vector that maps $\mathcal{R}$ to $\mathcal{R}'$.
$\mathcal{R}'$ is a region that contains a lattice point.



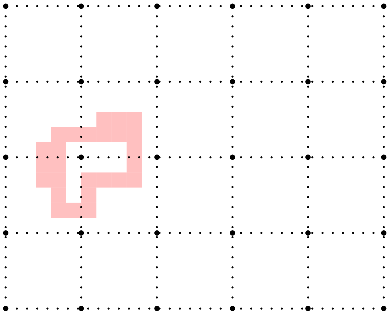
More than One Lattice Point

It appears that $\mathcal{R}$ could be translated to include two lattice points, but we want an algorithm rather than mere *guess and check*.



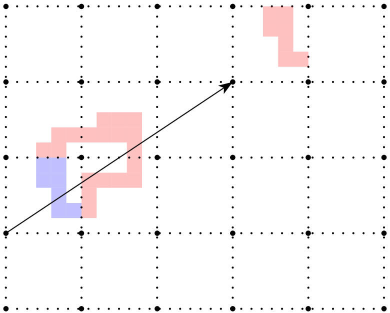
More than One Lattice Point

Use the lattice points to create a grid of lattice lines.



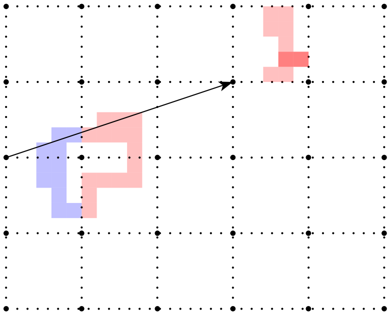
Stacking Tiles 1

Take one *tile* from $\mathcal{R}$ and translate it to a lattice square outside of $\mathcal{R}$.



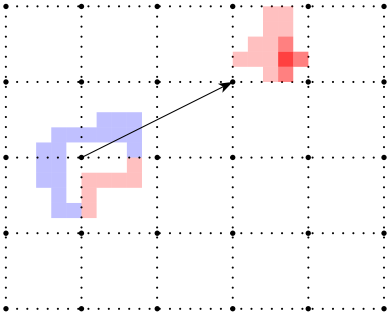
Stacking Tiles 2

Take another tile from $\mathcal{R}$ and translate it to the same lattice square outside of $\mathcal{R}$.
Notice how two pixels are a darker red than the others.



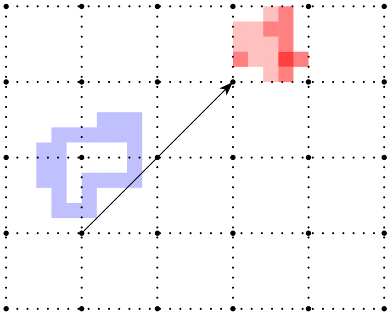
Stacking Tiles 3

Take the third tile from $\mathcal{R}$ and translate it to the same lattice square outside of $\mathcal{R}$.



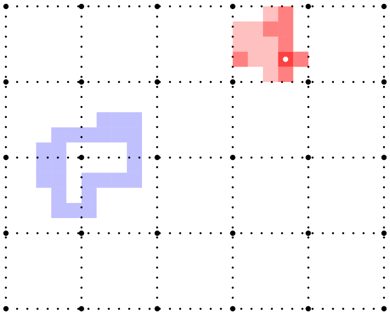
Stacking Tiles 4

Finally, take the last tile from $\mathcal{R}$ and translate it to the same lattice square outside of $\mathcal{R}$.
Notice we have an even darker red pixel where three red pixels are stacked.



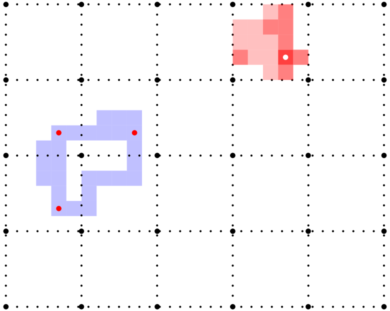
Stick in a Pin in the Stack

Place a pin through a position with multiple stacked pixels (points) and



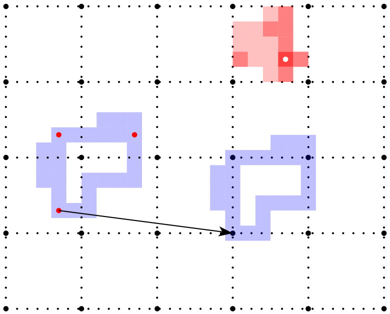
Stick in a Pin in the Stack

Place a pin through a position with multiple stacked pixels (points) and find the preimages in $\mathcal{R}$.



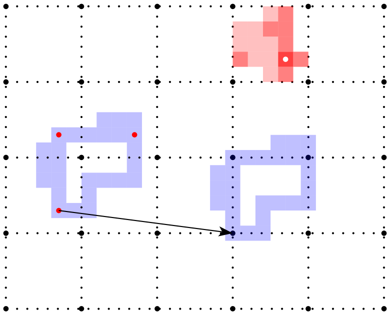
Translate Using a Preimage

Use a translation vector from any of the three preimages to an lattice point.



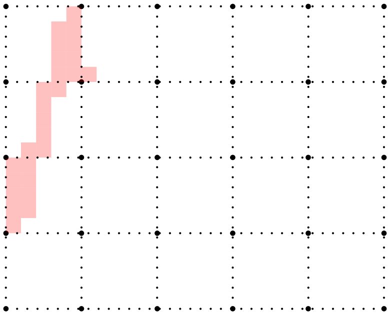
Assuring a Translation

What conditions do we need to be sure we can find a translation that maps a region to one that contains at least two lattice points?



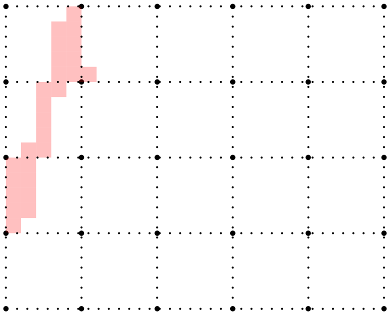
Looking at a Region with Area Greater than One a

Now consider a new region (say $\mathcal{S}$) whose area is slightly greater than 1.



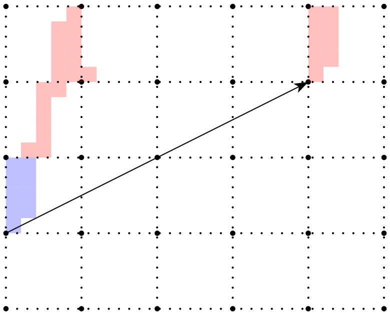
Looking at a Region with Area Greater than One a

Now consider a new region (say $\mathcal{S}$) whose area is slightly greater than 1.
Notice it does not contain any lattice points.
(Though they are close.)



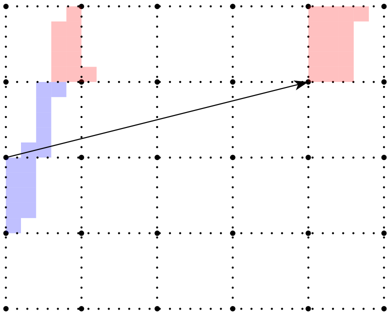
Looking at a Region with Area Greater than One b

Start our algorithm to find a translation (if it exists) that will yield at least two lattice points.



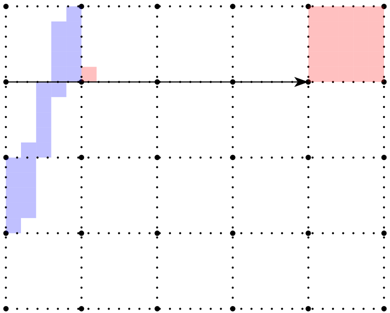
Looking at a Region with Area Greater than One c

Continue our algorithm.



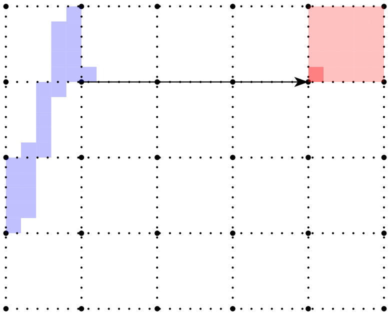
Looking at a Region with Area Greater than One d

Still, continue our algorithm.



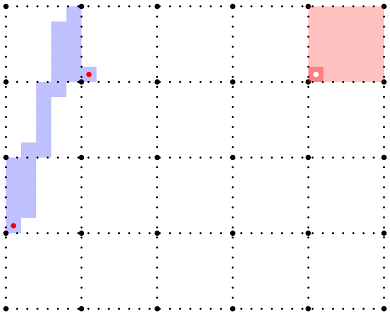
Looking at a Region with Area Greater than One e

Finish the last tile translation. Notice that we do indeed have a place with a stack of two pixels.



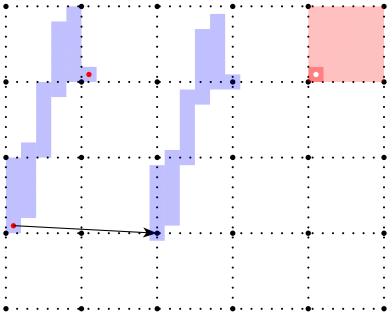
Looking at a Region with Area Greater than One f

Stick a pin into a darker pixel which corresponds to pixels in multiple tiles.



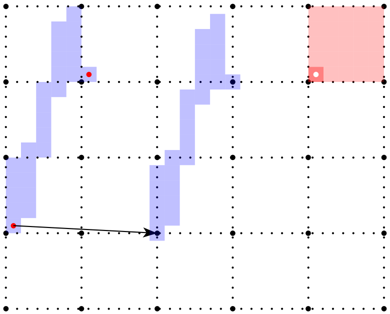
Looking at a Region with Area Greater than One g

Use a preimage of the pin to create a translation that maps the original region ($\mathcal{S}$ to $\mathcal{S}'$) that contains two lattice points.



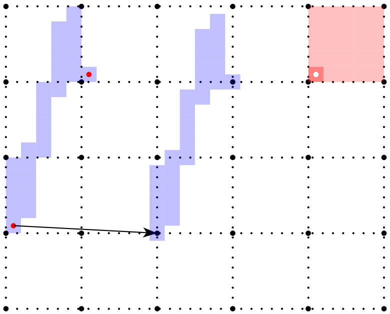
Containing vs Translated to Lattice Points

Although $\mathcal{S}$ does not contain any lattice points, it can be translated such that the image, $\mathcal{S}'$, contains two lattice points.



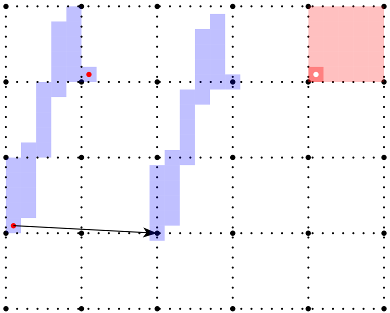
Blichfeldt's Lemma (for n=1)

Any plane region with an area in excess of 1 unit can be translated to a position where it covers at least 2 lattice points.



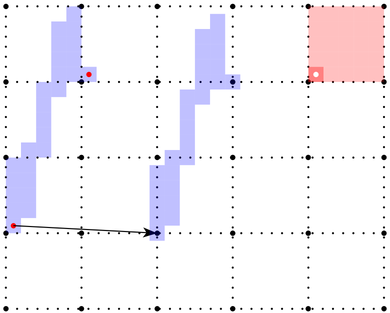
Blichfeldt's Lemma: Look for a translation

Consider our algorithm to find a translation. For any region with an area greater than one, at least one pixel position (point) in the receiving unit tile must be covered by at least two colored pixels (points).

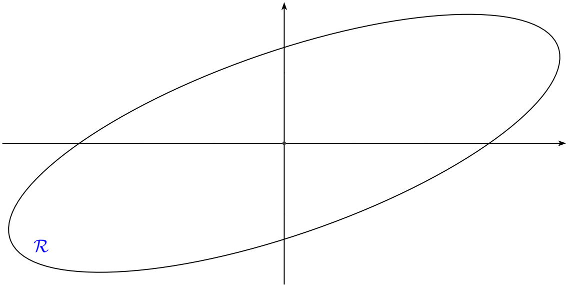


Blichfeldt's Lemma: It has to be there

If this were not the case,
it would contradict the
given that the region's
area is greater than one.

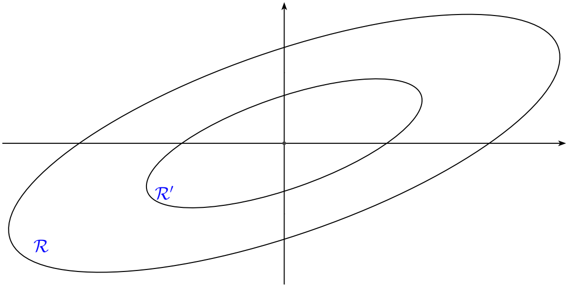


A Minkowski Region



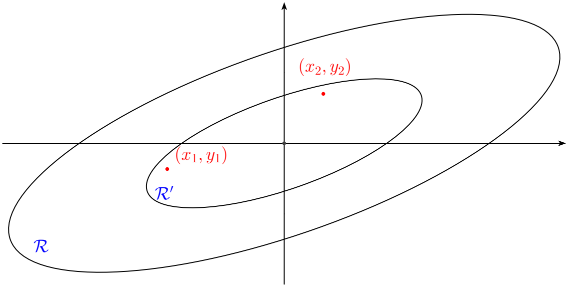
Let $\mathcal{R}$ be a convex plane region, symmetric about the origin with area exceeding 4.

A Minkowski Region: Dilated



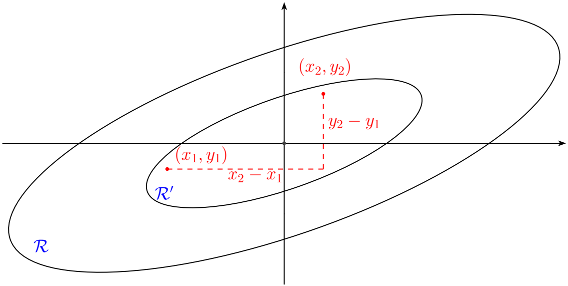
$\mathcal{R}'$ is the result of dilating $\mathcal{R}$ about the origin by a factor of $\frac{1}{2}$, i.e., $(x, y) \mapsto (\frac{x}{2}, \frac{y}{2})$. So $\mathcal{R}'$ is also convex, symmetric about the origin, and has an area one-fourth of $\mathcal{R}$. The area exceeds 1.

A Minkowski Region: Blichfeldt Points



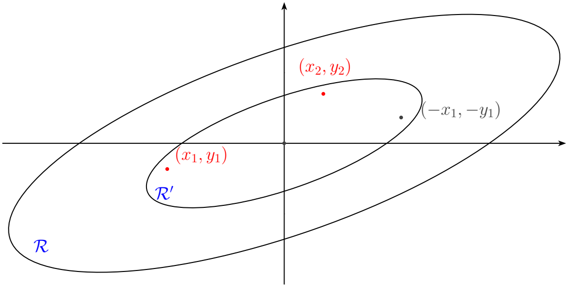
Having area exceeding 1, $\mathcal{R}'$ has a pair of points, P_1 at (x_1, y_1) and P_2 at (x_2, y_2)

A Minkowski Region: Blichfeldt Points



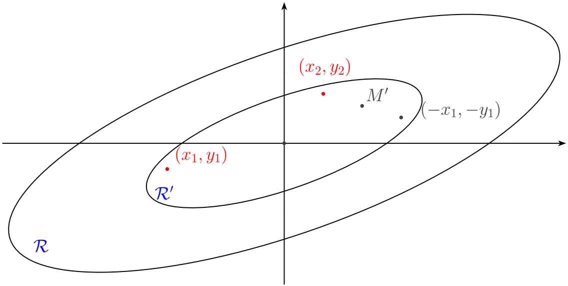
Having area exceeding 1, $\mathcal{R}'$ has a pair of points at (x_1, y_1) and at (x_2, y_2) such that $x_2 - x_1$ and $y_2 - y_1$ are integers.

A Minkowski Region: Use Symmetry



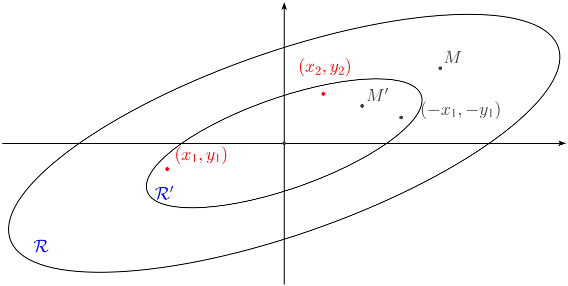
(x_1, y_1) has a partner across the origin at $(-x_1, -y_1)$ contained in $\mathcal{R}'$.

A Minkowski Region: Use Convexity



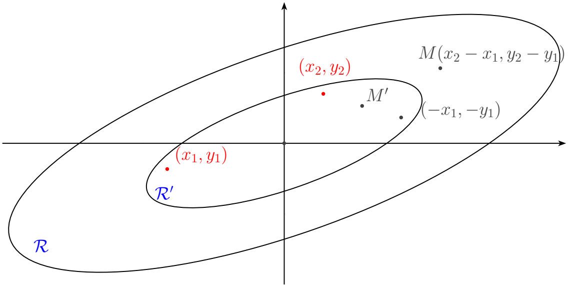
M', the midpoint of P_2 and P_1' is also contained in $\mathcal{R}'$, and has coordinates $(\frac{x_2 - x_1}{2}, \frac{y_2 - y_1}{2})$.

A Minkowski Region: Dilate Back



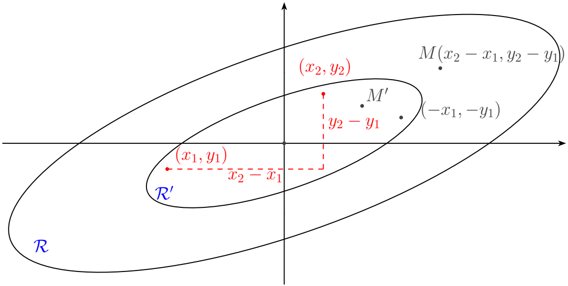
Since M' is at $(\dfrac{x_2 - x_1}{2}, \dfrac{y_2 - y_1}{2})$, M, the preimage of M' is at $(x_2 - x_1, y_2 - y_1)$,

A Minkowski Region: Dilate Back



Since M' is at $(\dfrac{x_2 - x_1}{2}, \dfrac{y_2 - y_1}{2})$, M, the preimage of M' is at $(x_2 - x_1, y_2 - y_1)$,

A Minkowski Region: Dilate Back



Since M' is at $(\dfrac{x_2 - x_1}{2}, \dfrac{y_2 - y_1}{2})$, M, the preimage of M' is at $(x_2 - x_1, y_2 - y_1)$, which are known integer coordinates, so M is indeed a lattice point in $\mathcal{R}$.

$\therefore$ any convex region with area in excess of 4 square units and symmetric about the origin must contain a lattice point other than the origin.