# Revisiting Numerical Integration:

## Getting More from Fewer Points

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## Romberg Integration

The method arises from a technique called Richardson extrapolation which can be used whenever the error $E(h)$ can be expanded in a series of the form

$$E(h) = c_1 h + c_2 h^2 + c_3 h^3 + c_4 h^4 + \cdots$$

We'll illustrate Richardson's technique by applying it to the trapezoidal rule.

$$I = \int_a^b f(x)\,dx = T(h) + E(h), \quad \text{with } E(h) = O\left(h^2\right)$$

In fact we can show that if $f$ can be expanded in a Taylor series on each subinterval then $E$ can be expanded in a series of the form

$$E(h) = c_2 h^2 + c_4 h^4 + c_6 h^6 + c_8 h^8 + \cdots$$

## Romberg Integration

To implement the method we don't need to know the coefficients, we need only know that they exist. Assume that we have computed trapezoidal estimates for $I$ for $h, h/2, h/4, h/8,$ so we have $T(h),\ T(h/2),\ T(h/4),\ T(h/8)$. Just looking at the first two we have

$$I \;=\; T(h) + E(h) \;=\; T(h) + c_2 h^2 + c_4 h^4 + c_6 h^6 + c_8 h^8 + \cdots$$

and $$I \;=\; T\left(\frac{h}{2}\right) + E\left(\frac{h}{2}\right) \;=\; T\left(\frac{h}{2}\right) + c_2\frac{h^2}{2^2} + c_4\frac{h^4}{2^4} + c_6\frac{h^6}{2^6} + c_8\frac{h^8}{2^8} + \cdots$$

We can eliminate the $h^2$ term by multiplying the second by $2^2$ and subtracting the first giving

$$\left(2^2-1\right)I \;=\; 2^2 T\left(\frac{h}{2}\right) - T(h) + c_4'\frac{h^4}{2^4} + c_6'\frac{h^6}{2^6} + c_8'\frac{h^8}{2^8} + \cdots$$

Which we can rewrite as

$$\left(2^2-1\right)I \;=\; \left(2^2-1\right)T\left(\frac{h}{2}\right) + T\left(\frac{h}{2}\right) - T(h) + c_4'\frac{h^4}{2^4} + c_6'\frac{h^6}{2^6} + c_8'\frac{h^8}{2^8} + \cdots$$

## Romberg Integration

The last equation is equivalent to

$$I \;=\; T\left(\frac{h}{2}\right) + \frac{T\left(\frac{h}{2}\right) - T(h)}{\left(2^2-1\right)} + k_4\frac{h^4}{2^4} + k_6\frac{h^6}{2^6} + k_8\frac{h^8}{2^8} + \cdots$$

Which is in the form $I \;=\; \text{estimate} \;+\; \text{error}$

$$=\qquad T_2(h) \qquad + k_4\frac{h^4}{2^4} + k_6\frac{h^6}{2^6} + k_8\frac{h^8}{2^8} + \cdots$$

$T_2(h)$ is an estimate for $I$ with an error that is $O(h^4)$

## Romberg Integration

We can repeat the process beginning with $I = T_2(h) + k_4\frac{h^4}{2^4} + k_6\frac{h^6}{2^6} + k_8\frac{h^8}{2^8} + \cdots$

And get a new estimate $I = T_3(h) + k_6'\frac{h^6}{2^6} + k_8'\frac{h^8}{2^8} + \cdots$ with

$$T_3(h) = T_2\left(\frac{h}{2}\right) + \frac{T_2\left(\frac{h}{2}\right) - T_2(h)}{\left(2^4 - 1\right)}$$

Continuing in this way we get a sequence of estimates for the integral $I$

$$T_1(h) = T(h), \quad T_j(h) = T_{j-1}\left(\frac{h}{2}\right) + \frac{T_{j-1}\left(\frac{h}{2}\right) - T_{j-1}(h)}{\left(2^{2(j-1)} - 1\right)}$$

Where for each estimate we have $I = T_j(h) + E_j(h)$, with $E_j(h) = O\left(h^{2j}\right)$

## Romberg Integration -- An example

To see how Romberg integration works we organize our estimates into an array as follows

$$\begin{array}{llll}
T_1(h) & & & \\
T_1\left(\frac{h}{2}\right) & T_2(h) & & \\
T_1\left(\frac{h}{4}\right) & T_2\left(\frac{h}{2}\right) & T_3(h) & \\
T_1\left(\frac{h}{8}\right) & T_2\left(\frac{h}{4}\right) & T_3\left(\frac{h}{2}\right) & T_4(h)
\end{array}$$

$$\boxed{T_{j-1}\left(\frac{h}{2}\right) + \frac{T_{j-1}\left(\frac{h}{2}\right) - T_{j-1}(h)}{\left(2^{2(j-1)} - 1\right)}}$$

The accuracy increases as we move down and as we move to the right.

Next we perform Romberg integration for a familiar example.

$$\int_1^3 x^{3/2}dx$$

We place our trapezoidal estimates in the first column. The first estimate corresponds to a partition of [1, 3] into 4 subintervals, an *h* value of ½. Accuracy improves as we move down the column

$$\int_1^3 x^{3/2}dx$$

$$= \frac{2}{5}x^{5/2}\Big|_1^3 = \frac{2}{5}\left(3^{5/2} - 1^{5/2}\right) = 5.835382907$$

| | | | |
|---|---|---|---|
| 5.858234 | | | |
| 5.841100 | 5.835388 | | |
| 5.836813 | 5.835384 | 5.835384 | |
| 5.835740 | 5.835382 | 5.835382 | 5.835382 |

# Romberg Integration -- An example

We can improve our Romberg result if we start with more precise (though quite inaccurate) trapezoidal estimates in the first column. The averaging method used to get the more accurate estimates involves the subtraction of nearly equal numbers which can lead to precision loss as the computation progresses. The distinction between precision and accuracy in computation is important in practical numerical analysis.

5.858234

5.841100 5.835388

5.836813 5.835384 5.835384

5.835740 5.835382 5.835382 5.835382

$\int_1^3 x^{3/2}dx$

$\approx 5.835382907$

5.85823385920

5.84110042206 5.83538927635

5.83681259147 5.835383314605.83538291715

5.83574034752 5.83538293287 5.835382907425.83538290726

*Note*: The correct value to 12 significant digits is 5.83538290725

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# Gaussian Quadrature

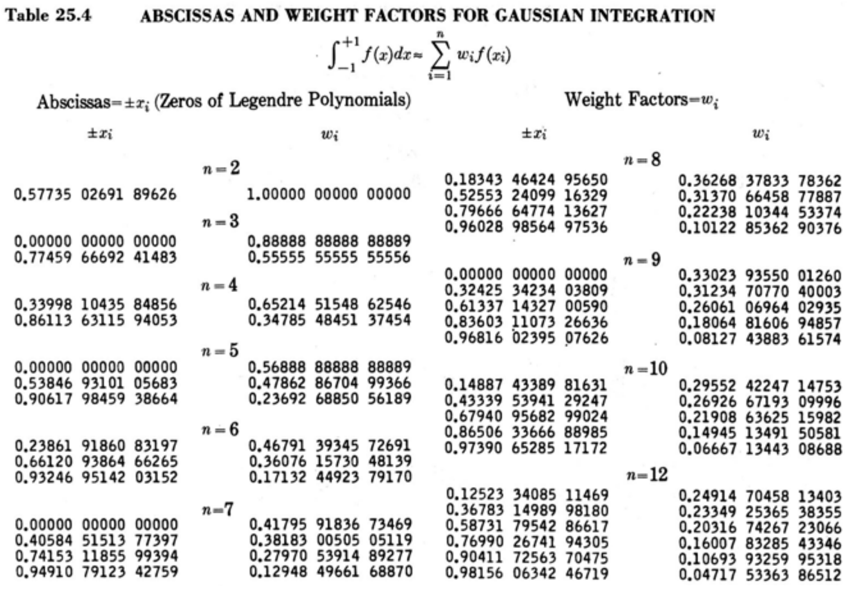

**Table 25.4 ABSCISSAS AND WEIGHT FACTORS FOR GAUSSIAN INTEGRATION**

$$\int_{-1}^{+1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

Abscissas $= \pm x_i$ (Zeros of Legendre Polynomials) Weight Factors $= w_i$

| $n$ | $\pm x_i$ | $w_i$ |
|---|---|---|
| $n=2$ | 0.57735 02691 89626 | 1.00000 00000 00000 |
| $n=3$ | 0.00000 00000 00000 | 0.88888 88888 88889 |
| | 0.77459 66692 41483 | 0.55555 55555 55556 |
| $n=4$ | 0.33998 10435 84856 | 0.65214 51548 62546 |
| | 0.86113 63115 94053 | 0.34785 48451 37454 |
| $n=5$ | 0.00000 00000 00000 | 0.56888 88888 88889 |
| | 0.53846 93101 05683 | 0.47862 86704 99366 |
| | 0.90617 98459 38664 | 0.23692 68850 56189 |
| $n=6$ | 0.23861 91860 83197 | 0.46791 39345 72691 |
| | 0.66120 93864 66265 | 0.36076 15730 48139 |
| | 0.93246 95142 03152 | 0.17132 44923 79170 |
| $n=7$ | 0.00000 00000 00000 | 0.41795 91836 73469 |
| | 0.40584 51513 77397 | 0.38183 00505 05119 |
| | 0.74153 11855 99394 | 0.27970 53914 89277 |
| | 0.94910 79123 42759 | 0.12948 49661 68870 |
| $n=8$ | 0.18343 46424 95650 | 0.36268 37833 78362 |
| | 0.52553 24099 16329 | 0.31370 66458 77887 |
| | 0.79666 64774 13627 | 0.22238 10344 53374 |
| | 0.96028 98564 97536 | 0.10122 85362 90376 |
| $n=9$ | 0.00000 00000 00000 | 0.33023 93550 01260 |
| | 0.32425 34234 03809 | 0.31234 70770 40003 |
| | 0.61337 14327 00590 | 0.26061 06964 02935 |
| | 0.83603 11073 26636 | 0.18064 81606 94857 |
| | 0.96816 02395 07626 | 0.08127 43883 61574 |
| $n=10$ | 0.14887 43389 81631 | 0.29552 42247 14753 |
| | 0.43339 53941 29247 | 0.26926 67193 09996 |
| | 0.67940 95682 99024 | 0.21908 63625 15982 |
| | 0.86506 33666 88985 | 0.14945 13491 50581 |
| | 0.97390 65285 17172 | 0.06667 13443 08688 |
| $n=12$ | 0.12523 34085 11469 | 0.24914 70458 13403 |
| | 0.36783 14989 98180 | 0.23349 25365 38355 |
| | 0.58731 79542 86617 | 0.20316 74267 23066 |
| | 0.76990 26741 94305 | 0.16007 83285 43346 |
| | 0.90411 72563 70475 | 0.10693 93259 95318 |
| | 0.98156 06342 46719 | 0.04717 53363 86512 |

## Gaussian Quadrature

In Gaussian quadrature the approximation to the integral is also a weighted sum of function values. The evaluation points (which are not equally spaced ) and weights are chosen to provide a very efficient estimate.

One way to implement Gaussian quadrature is to first transform the integral to an integral from –1 to 1, and then, for appropriate numbers of points *n*, get the gauss points and weights from the table. For our example we have

$$I = \int_1^3 x^{3/2} dx = \int_{-1}^1 (u+2)^{3/2}\, du \ , \qquad \text{using the linear substitution } u = x-2$$

So using $n = 5$ points

$$I = \int_{-1}^1 f(u)\, du = \int_{-1}^1 (u+2)^{3/2}\, du \approx \sum_{i=1}^{5} w_i f(u_i)$$

$$\begin{aligned} &\approx .23692885 f(-.906179846) + .478628671 f(-.538469310) \\ &\quad + .5688888889 f(0) + .23692885 f(.906179846) \\ &\quad + .478628671 f(.538469310) \end{aligned}$$

$\boxed{\approx 5.8353829 \text{ which is correct to 7 decimal places.}}$

## Bibliography

1. M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965
2. R. Burden, J. Faires, *Numerical Analysis*, 7th Edition, Brooks/Cole, 2001
3. F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956
4. http://numericalmethods.eng.usf.edu/anecdotes/romberg.doc
5. http://www-history.mcs.st-andrews.ac.uk/Biographies/
6. R Development Core Team, *R: A language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria, 2006, <http://www.R-project.org>
7. L. F. Richardson and J. L. Gaunt, *The deferred approach to the limit,* Philosophical Transactions of the Royal Society of London 226A, 1927
8. W. Romberg, *W. (1955), "Vereinfachte numerische Integration",* Norske Videnskabers Selskab Forhandlinger (Trondheim) ***28*** (7), 1955