The Gamma Function, Factorials and the Volumes of *n*-Balls

Wrapping up First Year Calculus in an *n*-Ball

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How to Reach us

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Our goal is to demonstrate to the students that the mathematics they learned this year has provided them with a very powerful problem solving tool.

We use the tools of first year calculus and some geometric intuition to derive a surprisingly simple formula for the volume of an n-dimensional ball.

Along the way we learn about the gamma function, the factorial function for non-integer values and some interesting trigonometric integrals.

What is Calculus?

I begin each school year with the idea that Calculus can be viewed from the perspective of three problems.

1. We know the slope of a line.
What is the “slope” of a curve at a point?

2. We know the “area under” a line segment.
What is the area under a curve?

3. We can compute values for functions that contain only addition, subtraction, multiplication, and division. How can we compute values for functions that use other operations?

Side Issues in Calculus

- Interpolating functions to increase the domain.

  e.g. $f\left(x\right) = 2^x$

  integers rationals reals

- New operations for function machines

  Arithmetic: $+, -, \bullet, \div$
  Powers and roots
  Exponential and log
  Trig

  Using Integrals:

  Accumulation Functions

  $$A(x) = \int_a^x f(t)dt$$

  Other parameters

  $$P(x) = \int_a^\infty f(x,t)dt$$

Interpolating functions to increase the domain

Consider $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$

Since $(n-1)! = (n-1) \cdot \ldots \cdot 2 \cdot 1$

we have $n! = n \cdot (n-1)!$

Note that nothing in this recursive definition demands *n* to be an integer. (except a starting value)

Perhaps we can extend the domain of factorials.

New operations for function machines:
The Gamma Function

We begin with a definition of the gamma function using one of the powerful new methods calculus provides for defining functions – the improper integral.

$$\Gamma(x) \;=\; \int_0^\infty t^{x-1}e^{-t}dt \qquad \text{for } \; x > 0$$

Then some "easy" exercises to gain familiarity

evaluate $\Gamma(1)$

graph $f(x,t) = t^{x-1}e^{-t}$ for $x = 0, \frac{1}{2}, 1, 2, 3, 4, 5$

The Gamma Function

Evaluate $\Gamma(1)$

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

$$\Gamma(1) = \int_0^\infty t^{1-1}e^{-t}dt$$

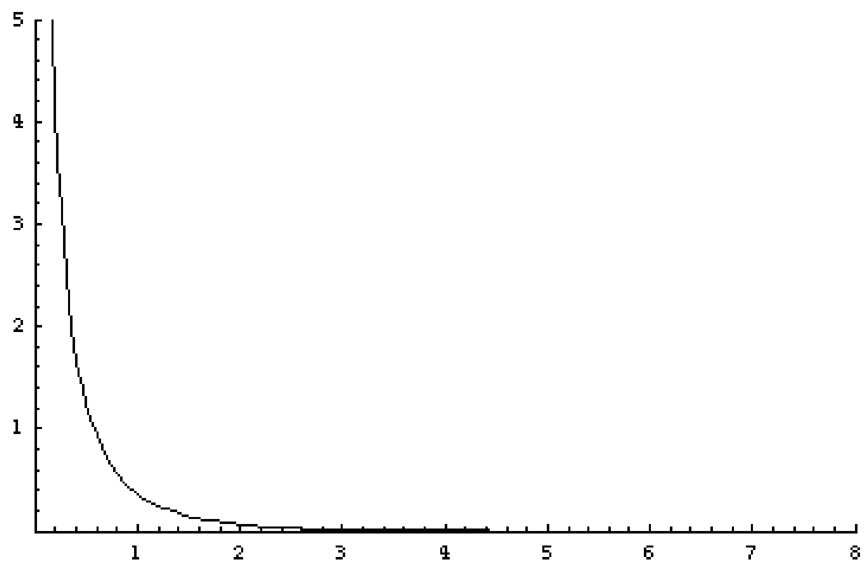
$$= \lim_{n\to\infty}\int_0^n e^{-t}dt$$

$$= \lim_{n\to\infty} -e^{-t}\Big|_0^n$$

$$= \lim_{n\to\infty} \frac{-1}{e^n} - \frac{-1}{e^0} = \lim_{n\to\infty} 1 - \frac{1}{e^n} = 1$$

Gamma Integrand

- x=0

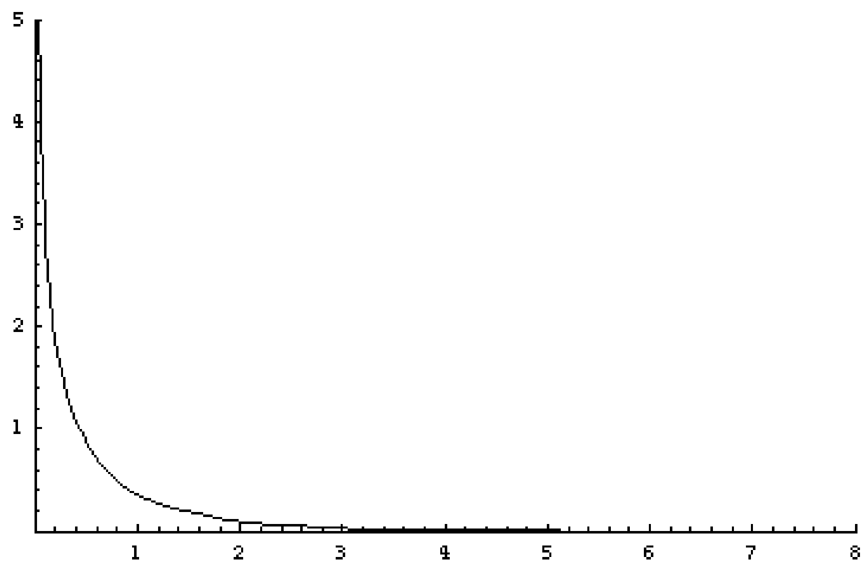


$$f(x,t) = t^{x-1}e^{-t}$$

Gamma Integrand

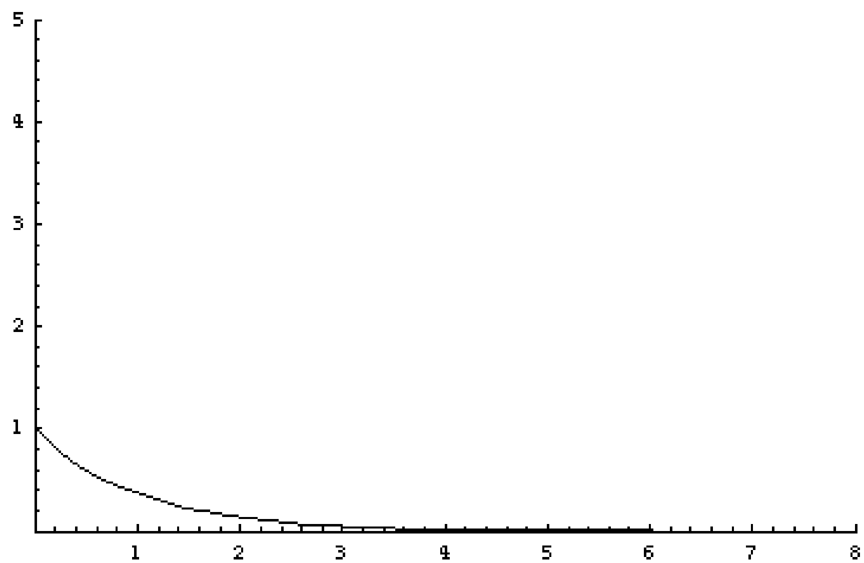
- x=1/2

$$f\left(x,t\right)=t^{x-1}e^{-t}$$



Gamma Integrand

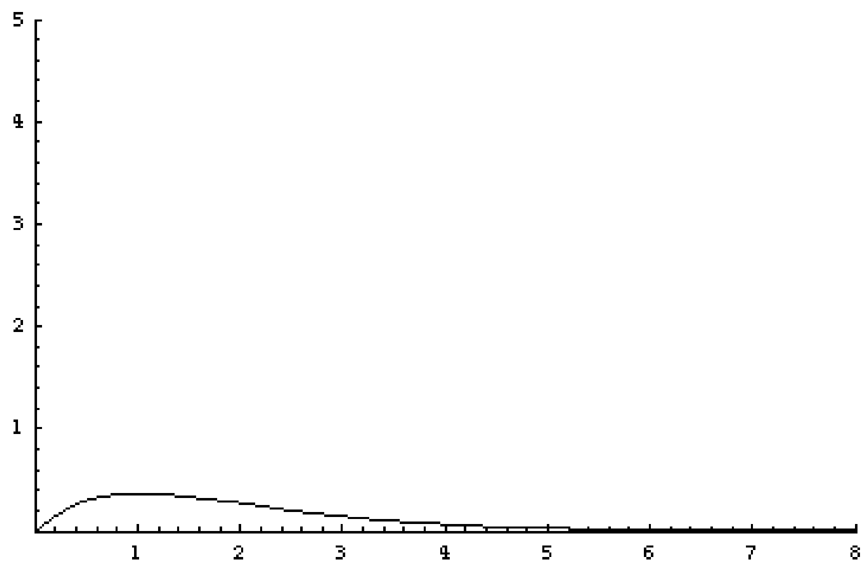
- x=1



$$f(x,t) = t^{x-1}e^{-t}$$

Gamma Integrand

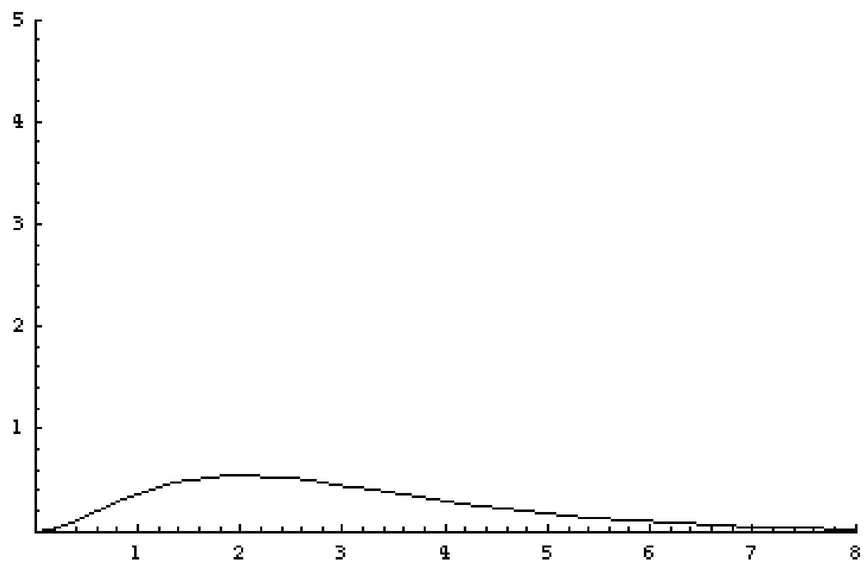
- x=2



$$f\left(x,t\right)=t^{x-1}e^{-t}$$

Gamma Integrand

- x=3

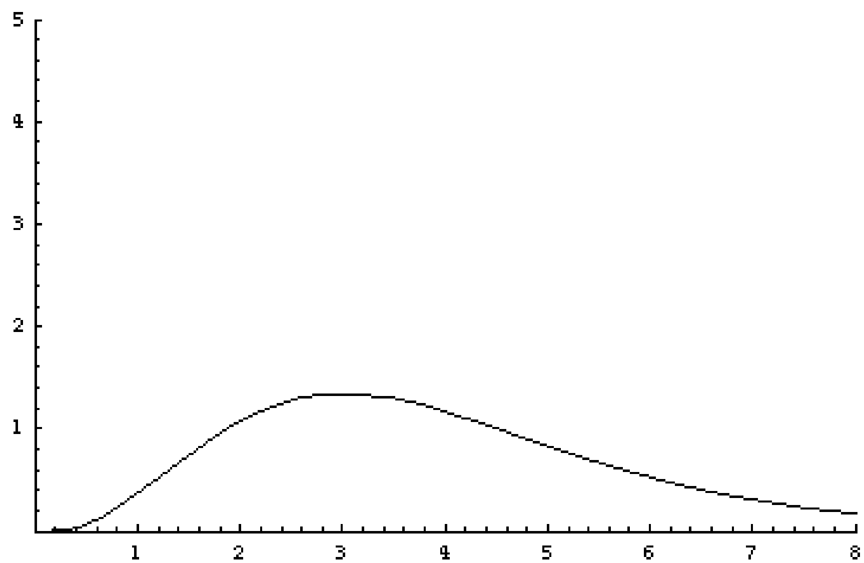


$$f(x,t) = t^{x-1}e^{-t}$$

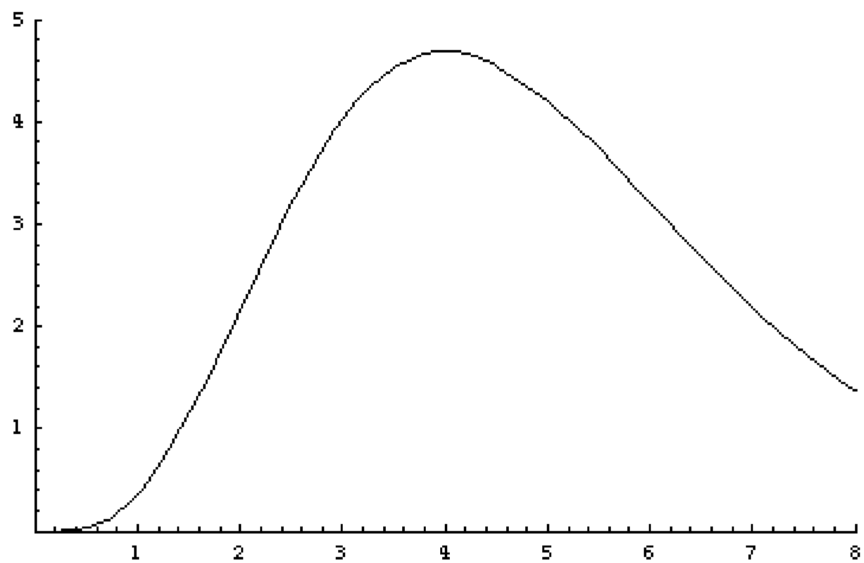
Gamma Integrand

- x=4

$$f(x,t) = t^{x-1}e^{-t}$$



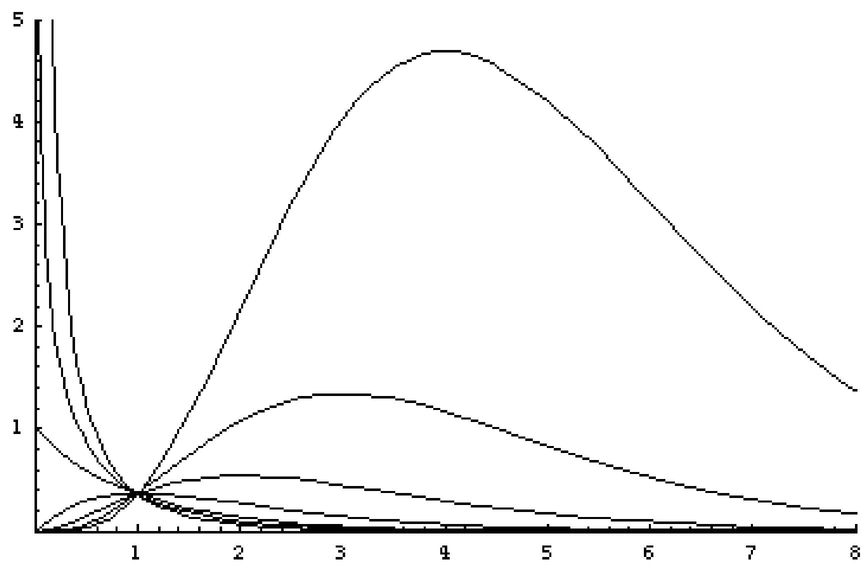
Gamma Integrand



- x=5

$$f(x,t) = t^{x-1}e^{-t}$$

Gamma Integrand



- x is 0, 1/2, 1, 2, 3, 4, 5

$$f\left(x,t\right)=t^{x-1}e^{-t}$$

The Gamma Function

Show $\Gamma(x+1) = x\Gamma(x)$

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt$$

Use integration by parts.

$$u = t^x \qquad dv = e^{-t} dt$$

$$du = xt^{x-1} dt \qquad v = -e^{-t}$$

$$\Gamma(x+1) = -t^x e^{-t}\Big|_0^\infty - \int_0^\infty \left(-e^{-t}\right) xt^{x-1} dt$$

$$\Gamma(x+1) = 0 + \int_0^\infty xt^{x-1} e^{-t} dt = x\int_0^\infty t^{x-1} e^{-t} dt$$

$$\therefore \Gamma(x+1) = x\Gamma(x)$$

The Gamma Function

How about $\Gamma(n)$ for integer values of n?

$$\Gamma(x+1) = x\Gamma(x) \qquad \Gamma(1) = 1$$

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1\cdot 1 = 1!$$

$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2\cdot 1 = 2!$$

$$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3\cdot 2! = 3!$$

$$\Gamma(5) = \Gamma(4+1) = 4\Gamma(4) = 4\cdot 3! = 4!$$

$$\vdots$$

$$\Gamma(n+1) = \Gamma(n+1) = n\Gamma(n) = n\cdot(n-1)! = n!$$

The Factorial Function

The result $\Gamma(n+1) = n!$ suggests a way to extend the domain of the factorial function beyond the non-negative integers.

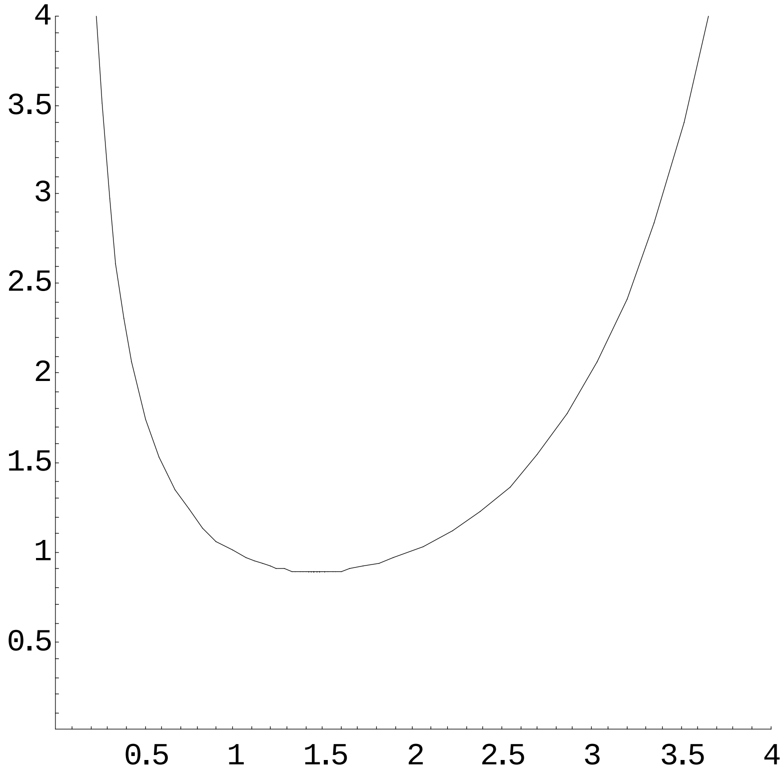
We define the factorial function for real $x > -1$ by

$$x! = \Gamma(x+1)$$

Before finding some new values of the factorial function we need one special value of the gamma function. We need to evaluate $\Gamma(1/2)$.

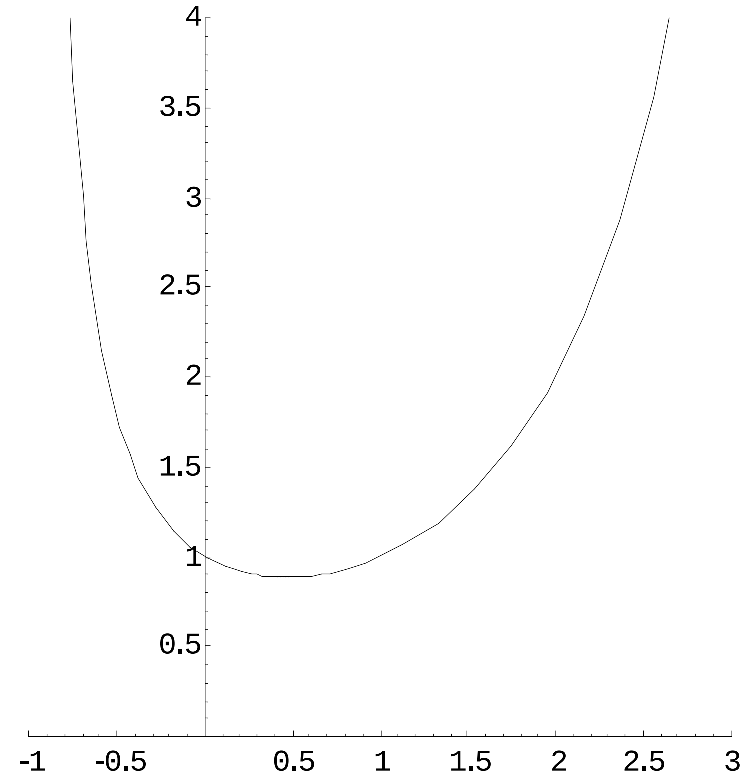
Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$



Factorial Function

$$x! = \Gamma(x+1)$$



$\Gamma\left(\tfrac{1}{2}\right)$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \Rightarrow$$

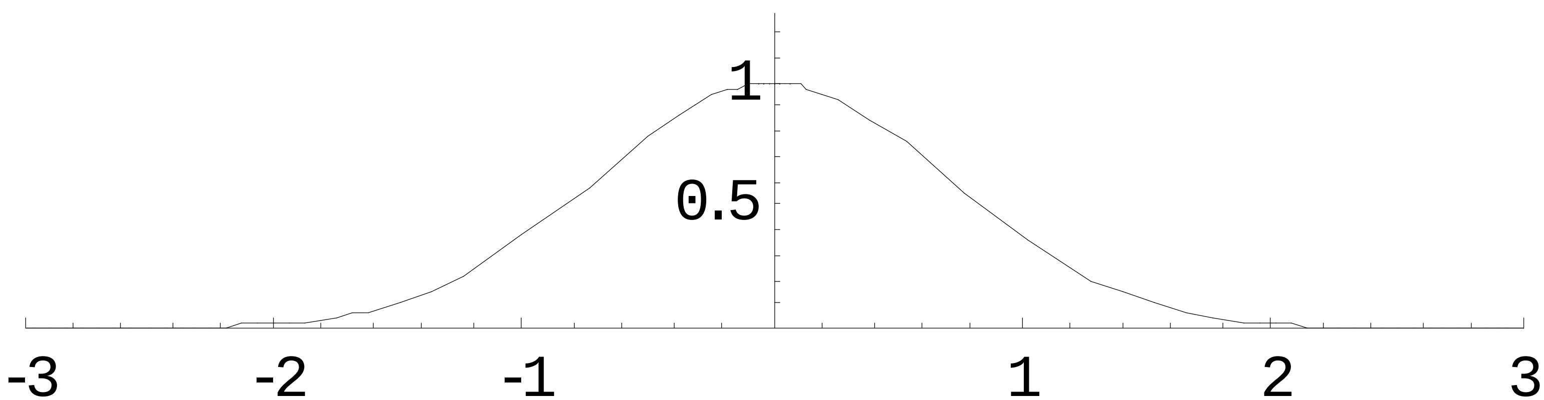
$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1} e^{-t} dt = \int_0^\infty t^{-1/2} e^{-t} dt$$

Letting $u = t^{1/2} \quad \Rightarrow \quad t = u^2, \quad du = (1/2)t^{-1/2} dt$

also u goes from 0 to ∞ as t goes from 0 to ∞

$$\int_0^\infty t^{-1/2} e^{-t} dt = \int_0^\infty e^{-u^2} 2du = 2\int_0^\infty e^{-u^2} du = \int_{-\infty}^\infty e^{-u^2} du$$

$$f(x) = e^{-x^2}$$



Two Views

"A mathematician is one to whom $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ is as obvious as that twice two makes four is to you."

Lord Kelvin

"Many things are not accessible to intuition at all, the value of $\int_{-\infty}^{\infty} e^{-x^2} dx$ for instance."

J. E. Littlewood

$\Gamma\left(\tfrac{1}{2}\right)$

$$\Gamma(1/2) = \int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$$

We have considered two options for trying to get the students to accept this:

1. a numerical exploration

2. a proof that follows from the properties of a sequence of integrals that we investigate later.

Factorials

We now go back to the definition $x! = \Gamma(x+1)$

Using the recursion $\Gamma(x+1) = x\Gamma(x)$ we can show that

$$x! = \Gamma(x+1) = x\Gamma(x) = x\Gamma\big((x-1)+1\big) = x(x-1)!$$

now with $x! = x(x-1)!$ and $\Gamma(1/2) = \sqrt{\pi}$ we can evaluate

$$\left(\frac{1}{2}\right)!,\ \left(\frac{2}{2}\right)!,\ \left(\frac{3}{2}\right)!,\ \left(\frac{4}{2}\right)!,\ \left(\frac{5}{2}\right)!,\ \left(\frac{6}{2}\right)!,\ \ldots,\ \left(\frac{n}{2}\right)!$$

Factorials

$$\left(\frac{1}{2}\right)! = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\left(\frac{2}{2}\right)! = (1)! = 1$$

$$\left(\frac{3}{2}\right)! = \left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)! = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)! = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$$

$$\left(\frac{4}{2}\right)! = \left(\frac{4}{2}\right)\left(\frac{2}{2}\right)! = \left(\frac{4}{2}\right)(1)$$

$$\left(\frac{5}{2}\right)! = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)! = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$$

$$\left(\frac{6}{2}\right)! = \left(\frac{6}{2}\right)\left(\frac{4}{2}\right)! = \left(\frac{6}{2}\right)(2)(1)$$

$$\left(\frac{7}{2}\right)! = \left(\frac{7}{2}\right)\left(\frac{5}{2}\right)! = \left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$$

$$\left(\frac{n}{2}\right)!$$

for n even $\left(\frac{n}{2}\right)! = \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots(3)(2)(1)$

an ordinary integer factorial

for n odd $\left(\frac{n}{2}\right)! = \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$

we march nicely down to $1/2$ and end with a $\sqrt{\pi}$

n-Space, n-Balls and n-Spheres

Euclidean n-dimensional space is the set of ordered Real n-tuples together with a Euclidean distance measure.

$$\mathbf{R}^n = \left\{ \left(x_1, x_2, \ldots, x_n \right) \mid x_1, x_2, \ldots, x_n \in \mathbf{R} \right\}$$

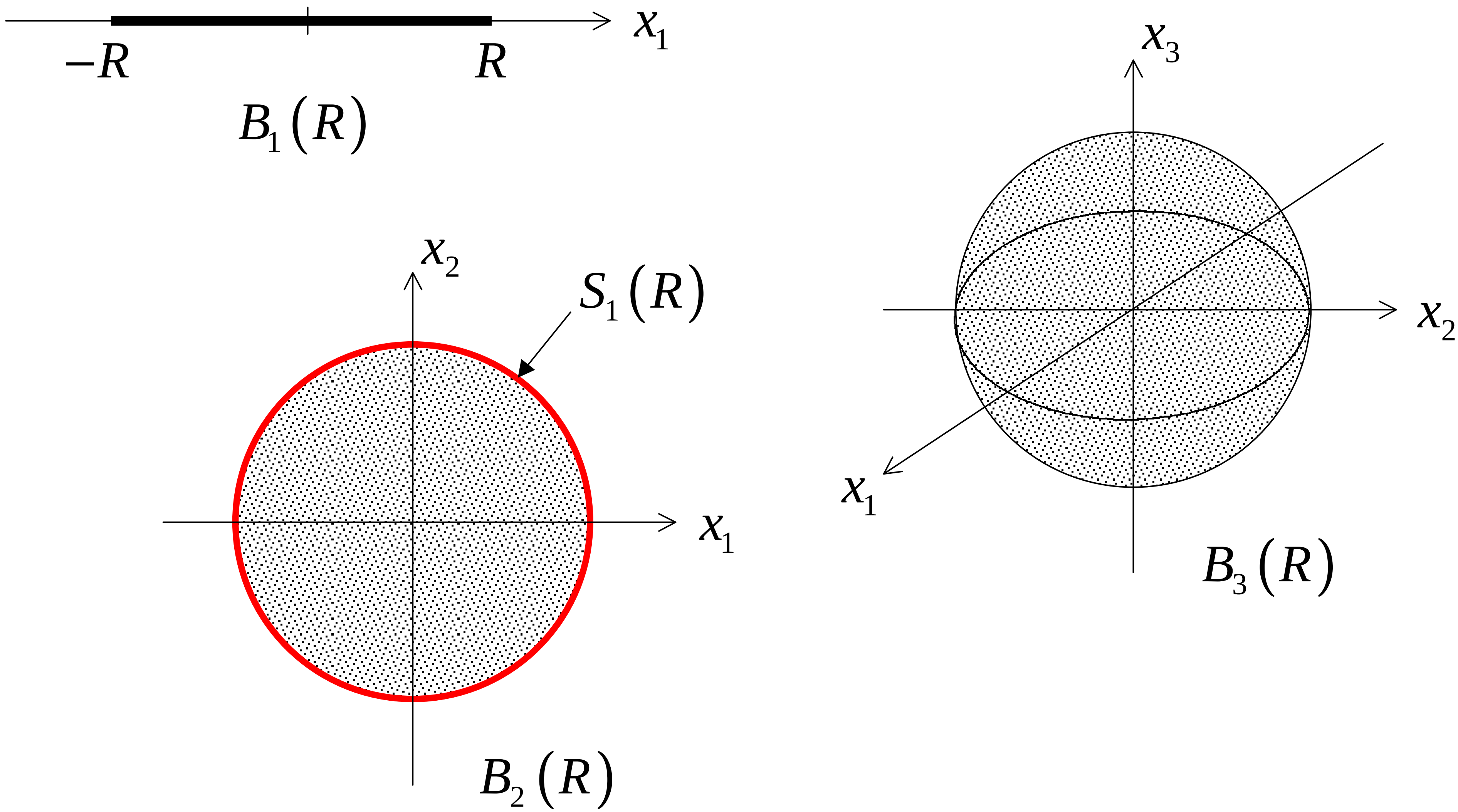
with the distance from the origin to $\left(x_1, \ldots, x_n \right) = \left(x_1^{\ 2} + x_2^{\ 2} + \cdots + x_n^{\ 2} \right)^{1/2}$

We define the n-ball, $B_n\left(R\right)$, of radius R and its boundary, $S_{n-1}\left(R\right)$ by

$$B_n\left(R\right) = \left\{ \left(x_1, x_2, \ldots, x_n \right) \in \mathbf{R}^n \mid \ x_1^{\ 2} + x_2^{\ 2} + \cdots + x_n^{\ 2} \le R^2 \right\}$$

$$S_{n-1}\left(R\right) = \left\{ \left(x_1, x_2, \ldots, x_n \right) \in \mathbf{R}^n \mid \ x_1^{\ 2} + x_2^{\ 2} + \cdots + x_n^{\ 2} = R^2 \right\}$$

1-ball, 2-ball, 3-ball, …



1-ball, 2-ball, 3-ball, …

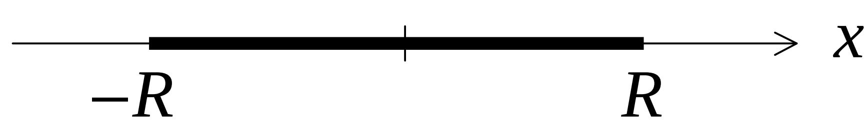
Some notation then some examples

let: $V_n(R) =$ The "n-volume" of $B_n(R)$

$A_{n-1}(R) =$ The "$n-1$ - volume" of $S_{n-1}(R)$

Examples

$n = 1$ $\qquad B_1(R) = \{x \in \mathbf{R} \mid x^2 \le R^2\} = \{x \in \mathbf{R} \mid -R \le x \le R\}$

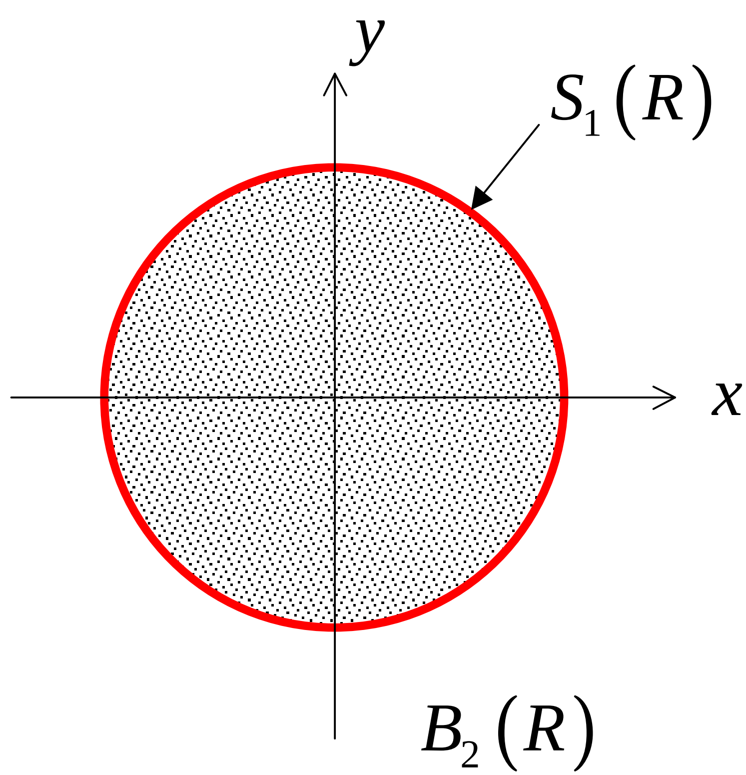


$V_1(R) = 2R \qquad A_0(R) = ?$

1-ball, 2-ball, 3-ball, …

Examples

$n = 2 \qquad B_2(R) = \left\{ (x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \le R^2 \right\}$

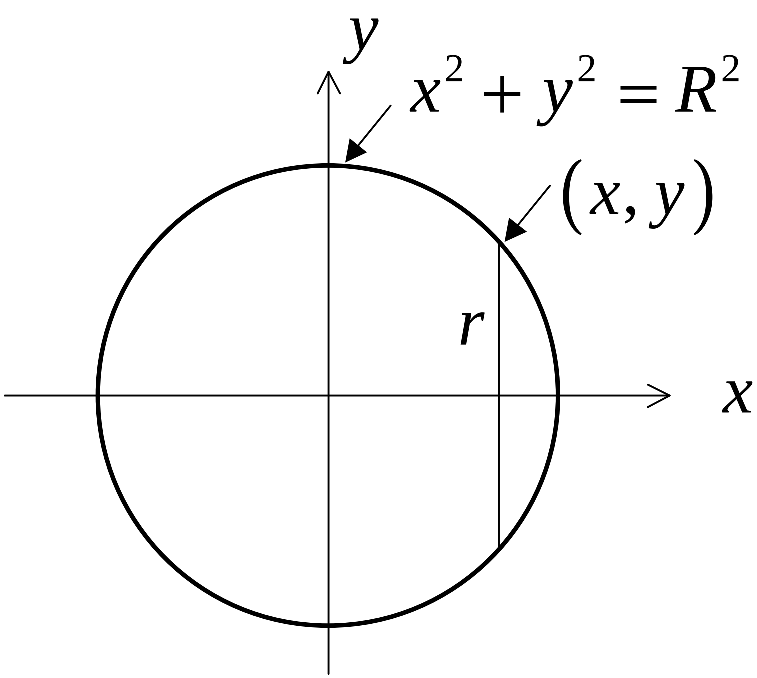


$V_2(R) = \pi R^2$

$A_1(R) = 2\pi R$

1-ball, 2-ball, 3-ball, …

Computing $V_2(R)$ using the "disk" method



$$\Delta V = 2r\Delta x$$

$$r = y = \sqrt{R^2 - x^2}$$

$$dV = 2\sqrt{R^2 - x^2}dx$$

$$V_2(R) = \int_{-R}^{R} dV = \int_{-R}^{R} 2\sqrt{R^2 - x^2}dx$$

$$= 2\int_{-R}^{R} \sqrt{R^2 - x^2}dx = 4\int_{0}^{R} \sqrt{R^2 - x^2}dx$$

using the trigonometric substitution

$$x = R\sin\theta, \quad dx = R\cos\theta d\theta$$

$$R^2 - x^2 = R^2\left(1 - \sin^2\theta\right) = R^2\cos^2\theta$$

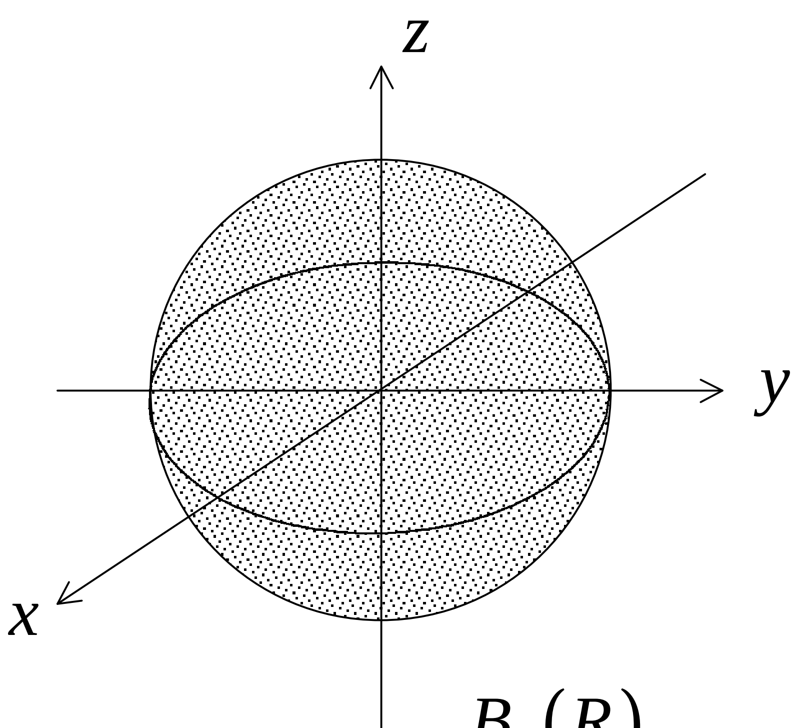
x goes from 0 to R as θ goes from 0 to $\frac{\pi}{2}$

$$V_2(R) = 4\int_0^{\frac{\pi}{2}} R^2\cos^2\theta d\theta = 4R^2\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$$

1-ball, 2-ball, 3-ball, …

Examples

$n = 3$ $\quad B_3(R) = \left\{ (x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 \le R^2 \right\}$



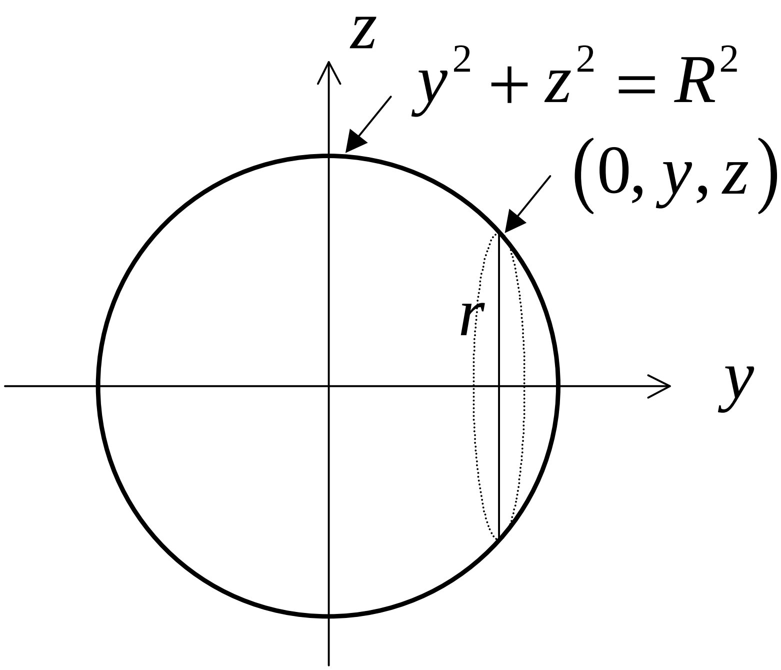
$B_3(R)$

$$V_3(R) = \frac{4}{3}\pi R^3$$

$$A_2(R) = 4\pi R^2$$

1-ball, 2-ball, 3-ball, …

Computing $V_3(R)$ using the disk method



$$\Delta V = \pi r^2 \Delta y$$

$$r = z = \sqrt{R^2 - y^2}$$

$$dV = \pi\left(R^2 - y^2\right)dy$$

$$V_3(R) = \int_{-R}^{R} dV = \int_{-R}^{R} \pi\left(R^2 - y^2\right)dy$$

$$= 2\pi\int_0^R \left(R^2 - y^2\right)dy$$

using the trigonometric substitution

$$y = R\sin\theta, \quad dy = R\cos\theta\, d\theta$$

$$R^2 - y^2 = R^2\left(1-\sin^2\theta\right) = R^2\cos^2\theta$$

y goes from 0 to R as θ goes from 0 to $\frac{\pi}{2}$

$$V_3(R) = 2\pi R^3 \int_0^{\frac{\pi}{2}} \cos^3\theta\, d\theta$$

recall $V_2(R) = 4R^2 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$

1-ball, 2-ball, 3-ball, …

n	$V_n(R)$	$A_{n-1}(R)$
0	?	?
1	$2R$	?
2	πR^2	$2\pi R$
3	$\frac{4\pi}{3}R^3$	$4\pi R^2$
4	?	?
⋮	⋮	⋮

1-ball, 2-ball, 3-ball, 4-ball,…

Examples

$n = 4$ $\qquad B_4(R) = \{(x, y, z, w) \in \mathbf{R}^4 \mid x^2 + y^2 + z^2 + w^2 \le R^2\}$

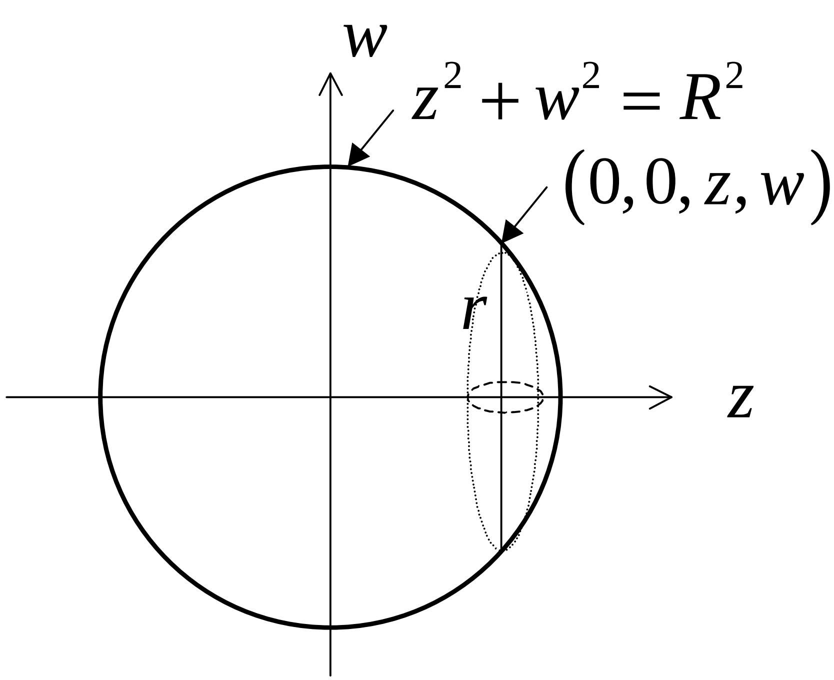
?

$$V_4(R) = ?\,R^4$$

$$A_3(R) = ??\,R^3$$

1-ball, 2-ball, 3-ball,4-ball, …

Computing $V_4(R)$ using the disk method



When we slice the 4-ball at a fixed value of z

$$\left\{(x,y,z,w) \in \mathbf{R}^4 \mid x^2 + y^2 + z^2 + w^2 \le R^2\right\}$$

reduces to

$$\left\{(x,y,z,w) \mid x^2 + y^2 + w^2 \le R^2 - z^2 = r^2\right\}.$$

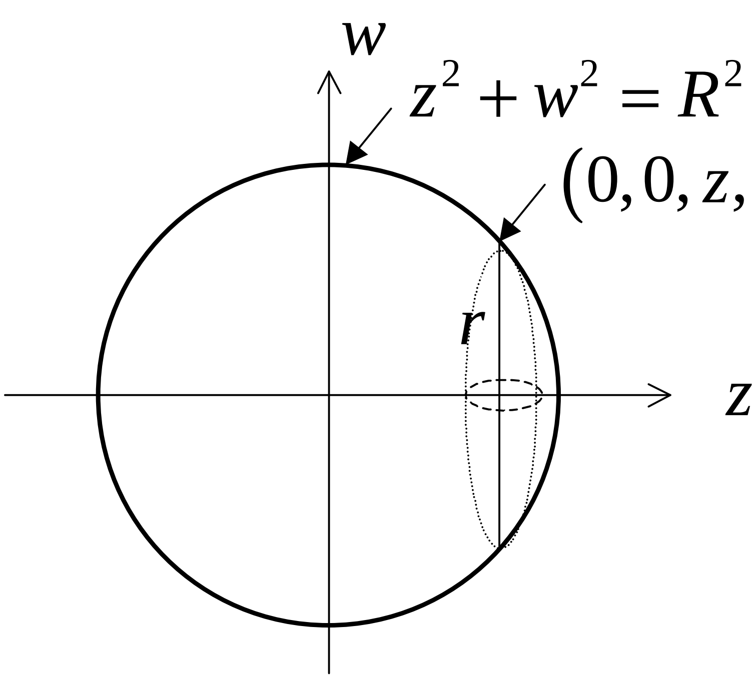
$$** \quad \Delta V = \frac{4}{3}\pi r^3 \Delta z$$

$$r = w = \sqrt{R^2 - z^2}$$

$$dV = \frac{4}{3}\pi\left(\sqrt{R^2 - z^2}\right)^3 dz$$

1-ball, 2-ball, 3-ball,4-ball, …

Computing $V_4(R)$ using the disk method



$$V_4(R) = \int_{-R}^{R} dV = \int_{-R}^{R} \frac{4}{3}\pi\left(\sqrt{R^2 - z^2}\right)^3 dz$$

$$= \frac{8}{3}\pi \int_0^R \left(\sqrt{R^2 - z^2}\right)^3 dz$$

using the trigonometric substitution

$$z = R\sin\theta, \quad dz = R\cos\theta\, d\theta$$

$$R^2 - z^2 = R^2\left(1 - \sin^2\theta\right) = R^2\cos^2\theta$$

z goes from 0 to R as θ goes from 0 to $\frac{\pi}{2}$

$$V_4(R) = \frac{8}{3}\pi R^4 \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

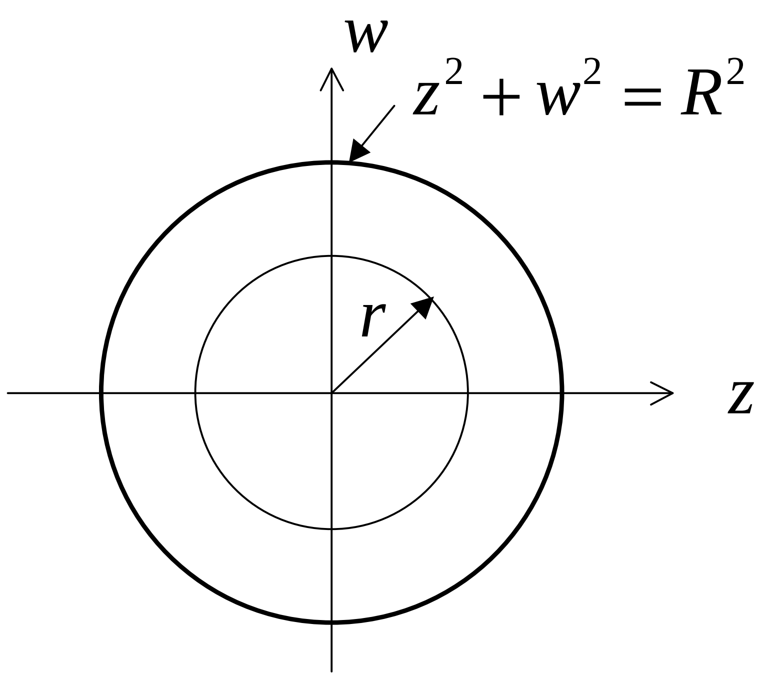
** $\Delta V = \frac{4}{3}\pi r^3 \Delta z$

$$r = w = \sqrt{R^2 - z^2}$$

$$dV = \frac{4}{3}\pi\left(\sqrt{R^2 - z^2}\right)^3 dz$$

1-ball, 2-ball, 3-ball,4-ball, …

Computing $V_4(R)$ using the shell method



$$V_4(R) = \int_0^R dV = \int_0^R A_3(r)\,dr$$

$V_4(R) = \int_0^R A_3(r)\,dr$ is in the form

$F(x) = \int_0^x f(t)\,dt$ hence by the

Fundamental Theorem of Calculus

$$\frac{d}{dR}(V_4) = V_4'(R) = A_3(R)$$

** $\Delta V = A_3(r)\Delta r$

$$dV = A_3(r)\,dr$$

… , n-ball, …

$$B_n(R) = \left\{ (x_1, \ldots, x_{n-1}, x_n) \in \mathbf{R}^n \mid x_1^{\,2} + \cdots + x_{n-1}^{\,2} + x_n^{\,2} \leq R^2 \right\}$$

?

$$V_n(R) = ?\, R^n$$

$$A_{n-1}(R) = ??\, R^{n-1}$$

We can be more explicit

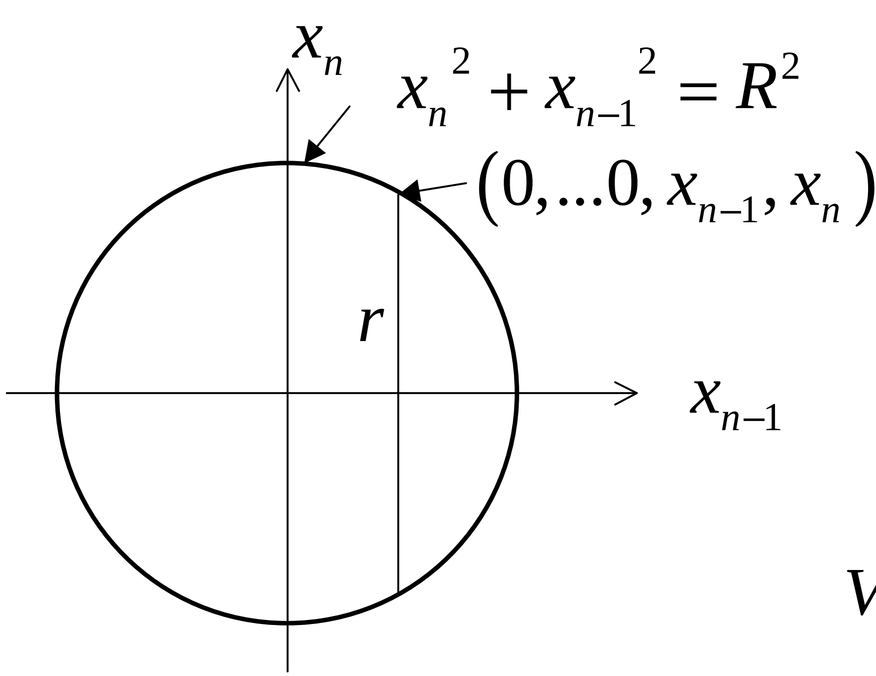
$$V_n(R) = c_n R^n$$

$$A_{n-1}(R) = V_n'(R) = n c_n R^{n-1}$$

1-ball, 2-ball, 3-ball, 4-ball, … , n-ball, …

n	$V_n(R)$	$A_{n-1}(R)$
0	?	?
1	$2R$	2
2	πR^2	$2\pi R$
3	$\frac{4\pi}{3}R^3$	$4\pi R^2$
4	$c_4 R^4$	$4c_4 R^3$
•	•	•
•	•	•
•	•	•
n	$c_n R^n$	$nc_n R^{n-1}$

… , n-ball, …



again using the disk method

$$\Delta V = V_{n-1}(r)\Delta x_{n-1} = V_{n-1}(x_n)\Delta x_{n-1}$$

$$r = x_n = \sqrt{R^2 - x_{n-1}{}^2}$$

$$dV = V_{n-1}\left(\sqrt{R^2 - x_{n-1}{}^2}\right)dx_{n-1}$$

$$V_n(R) = \int_{-R}^{R} dV = \int_{-R}^{R} V_{n-1}\left(\sqrt{R^2 - x_{n-1}{}^2}\right)dx_{n-1}$$

$$= 2\int_0^R V_{n-1}\left(\sqrt{R^2 - x_{n-1}{}^2}\right)dx_{n-1}$$

… , n-ball, …

$$V_n(R) = \int_{-R}^{R} dV = 2\int_0^R V_{n-1}(r)\,dx_{n-1} = 2\int_0^R V_{n-1}\left(\sqrt{R^2 - {x_{n-1}}^2}\right)dx_{n-1}$$

** letting $V_n(r) = c_n r^n$, we have $V_{n-1}(r) = c_{n-1} r^{n-1}$ and

$$V_n(R) = 2\int_0^R V_{n-1}\left(\sqrt{R^2 - {x_{n-1}}^2}\right)dx_{n-1} = 2c_{n-1}\int_0^R \left(\sqrt{R^2 - {x_{n-1}}^2}\right)^{n-1} dx_{n-1}$$

now using the trigonometric substitution $x_{n-1} = R\sin\theta$ implies $dx_{n-1} = R\cos\theta\, d\theta$ and $R^2 - {x_{n-1}}^2 = R^2\left(1 - \sin^2\theta\right) = R^2\cos^2\theta$.

So $V_n(R) = 2c_{n-1}R^n \int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = 2c_{n-1} I_n R^n$, where $I_n = \int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta$

Hence $V_n(R) = c_n R^n = 2c_{n-1} I_n R^n$, which implies

$$\boxed{c_n = 2c_{n-1} I_n}$$

Some Interesting and useful Integrals

Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta$

Show $I_0 = \frac{\pi}{2}, \; I_1 = 1$

Show $\int \cos^n \theta \, d\theta = \frac{1}{n} \sin\theta \cos^{n-1}\theta + \frac{n-1}{n} \int \cos^{n-2} \theta \, d\theta$

Show $\int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} \theta \, d\theta$

Hence $I_n = \frac{n-1}{n} I_{n-2}$

I_n for even and odd n

$$I_0 = \frac{\pi}{2}, \quad I_1 = 1 \quad \text{and} \quad I_n = \frac{n-1}{n} I_{n-2}$$

$$I_0 = \frac{\pi}{2}$$

$$I_2 = \frac{1}{2} I_0 = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I_4 = \frac{3}{4} I_2 = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I_6 = \frac{5}{6} I_4 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\vdots$$

$$I_{2k} = \frac{2k-1}{2k} \cdot \frac{2k-3}{2(k-1)} \cdot \frac{2k-5}{2(k-2)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I_1 = 1$$

$$I_3 = \frac{2}{3} I_1 = \frac{2}{3}$$

$$I_5 = \frac{4}{5} I_3 = \frac{4}{5} \cdot \frac{2}{3}$$

$$I_7 = \frac{6}{7} I_5 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

$$\vdots$$

$$I_{2k+1} = \frac{2k}{2k+1} \cdot \frac{2(k-1)}{2k-1} \cdot \frac{2(k-2)}{2k-3} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

Back to the n-ball

$$V_n(R) = c_n R^n, \quad c_n = 2c_{n-1} I_n \quad \text{also} \quad V_1(R) = 2R \quad \Rightarrow \quad c_1 = 2$$

$$c_2 = 2c_1 I_2 = 2(2)\left(\frac{\pi}{4}\right) = \pi$$

$$c_3 = 2c_2 I_3 = 2(\pi)\left(\frac{2}{3}\right) = \frac{4}{3}\pi$$

$$c_4 = 2c_3 I_4 = 2\left(\frac{4}{3}\pi\right)\left(\frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right) = \frac{1}{2}\pi^2$$

$$c_5 = 2c_4 I_5 = 2\left(\frac{1}{2}\pi^2\right)\left(\frac{4}{5}\frac{2}{3}\right) = \frac{8}{5\cdot 3}\pi^2$$

$$c_6 = 2c_5 I_6 = 2\left(\frac{4}{5}\frac{2}{3}\pi^2\right)\left(\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right) = \frac{1}{3\cdot 2}\pi^3$$

$$c_7 = 2c_6 I_7 = 2\left(\frac{1}{6}\pi^3\right)\left(\frac{6}{7}\frac{4}{5}\frac{2}{3}\right) = \frac{16}{7\cdot 5\cdot 3}\pi^3$$

$$c_8 = \frac{1}{4\cdot 3\cdot 2}\pi^4$$

$$c_9 = \frac{2^5}{9\cdot 7\cdot 5\cdot 3}\pi^4$$

$$c_{10} = \frac{1}{5\cdot 4\cdot 3\cdot 2}\pi^5$$

$$\vdots$$

$$c_n = \begin{cases} \dfrac{\pi^{n/2}}{\left(n/2\right)!} & \text{if } n \text{ is even} \\[2ex] \dfrac{??}{???} & \text{if } n \text{ is odd} \end{cases}$$

Back to the n-ball

$$c_n = \frac{\pi^{n/2}}{\left(n/2\right)!} \quad \text{if } n \text{ is even}$$

What about n odd ?

$$c_1 = 2$$

$$c_3 = \frac{4}{3}\pi$$

$$c_5 = \frac{8}{5\cdot 3}\pi^2$$

$$c_7 = \frac{2^4}{7\cdot 5\cdot 3}\pi^3$$

$$c_9 = \frac{2^5}{9\cdot 7\cdot 5\cdot 3}\pi^4 = \frac{\pi^4\sqrt{\pi}}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{\pi^{9/2}}{\left(9/2\right)!}$$

$\cdot$

$\cdot$

$\cdot$

$$c_n = \frac{\pi^{n/2}}{\left(n/2\right)!} \quad \text{if } n \text{ is odd so}$$

$$c_n = \frac{\pi^{n/2}}{\left(n/2\right)!} \quad \text{if } n \text{ is even or odd}$$

Finally

$$V_n(R) = c_n R^n = \frac{\pi^{n/2}}{\left(n/2\right)!} R^n$$

$$A_{n-1}(R) = n \frac{\pi^{n/2}}{\left(n/2\right)!} R^{n-1}$$

n	$V_n(R)$	$A_{n-1}(R)$
0	1	
1	$2R$	2
2	πR^2	$2\pi R$
3	$\frac{4\pi}{3} R^3$	$4\pi R^2$
4	$\frac{\pi^2}{2} R^4$	$2\pi^2 R^3$
5	$\frac{8\pi^2}{5} R^5$	$8\pi^2 R^4$
6	$\frac{\pi^3}{6} R^6$	$\pi^3 R^5$
⋮	⋮	⋮

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