

# The Gamma Function, Factorials and the Volumes of $n$ -Balls

Wrapping up First Year Calculus in an  $n$ -Ball

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## How to Reach us

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Our goal is to demonstrate to the students that the mathematics they learned this year has provided them with a very powerful problem solving tool.

We use the tools of first year calculus and some geometric intuition to derive a surprisingly simple formula for the volume of an n-dimensional ball.

Along the way we learn about the gamma function, the factorial function for non-integer values and some interesting trigonometric integrals.

## What is Calculus?

I begin each school year with the idea that Calculus can be viewed from the perspective of three problems.

1. We know the slope of a line.  
What is the “slope” of a curve at a point?
2. We know the “area under” a line segment.  
What is the area under a curve?
3. We can compute values for functions that contain only addition, subtraction, multiplication, and division. How can we compute values for functions that use other operations?

## Side Issues in Calculus

- Interpolating functions to increase the domain.

e.g.  $f(x) = 2^x$

integers      rationals      reals

- New operations for function machines

Arithmetic: +, -, •, ÷

Powers and roots

Exponential and log

Trig

Using Integrals:

Accumulation Functions

$$A(x) = \int_a^x f(t) dt$$

Other parameters

$$P(x) = \int_a^\infty f(x, t) dt$$

## Interpolating functions to increase the domain

Consider  $n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1$

Since  $(n-1)! = (n-1) \cdot \dots \cdot 2 \cdot 1$

we have  $n! = n \cdot (n-1)!$

Note that nothing in this recursive definition demands  $n$  to be an integer. (except a starting value)

Perhaps we can extend the domain of factorials.

## New operations for function machines: The Gamma Function

We begin with a definition of the gamma function using one of the powerful new methods calculus provides for defining functions – the improper integral.

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad \text{for } x > 0$$

Then some “easy” exercises to gain familiarity

evaluate  $\Gamma(1)$

graph  $f(x, t) = t^{x-1} e^{-t}$  for  $x = 0, \frac{1}{2}, 1, 2, 3, 4, 5$

## The Gamma Function

Evaluate  $\Gamma(1)$

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

$$\Gamma(1) = \int_0^{\infty} t^{1-1} e^{-t} dt$$

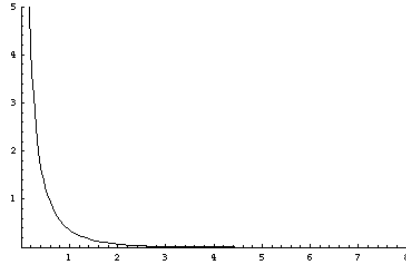
$$= \lim_{n \rightarrow \infty} \int_0^n e^{-t} dt$$

$$= \lim_{n \rightarrow \infty} -e^{-t} \Big|_0^n$$

$$= \lim_{n \rightarrow \infty} \frac{-1}{e^n} - \frac{-1}{e^0} = \lim_{n \rightarrow \infty} 1 - \frac{1}{e^n} = 1$$

## Gamma Integrand

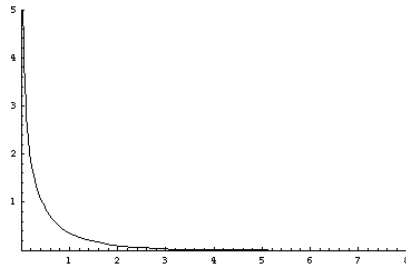
- $x=0$



$$f(x, t) = t^{x-1} e^{-t}$$

## Gamma Integrand

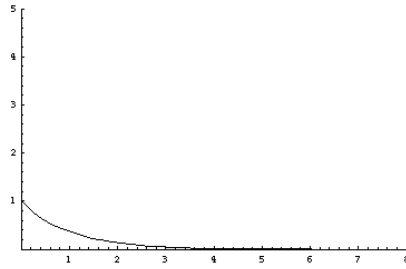
- $x=1/2$



$$f(x, t) = t^{x-1} e^{-t}$$

## Gamma Integrand

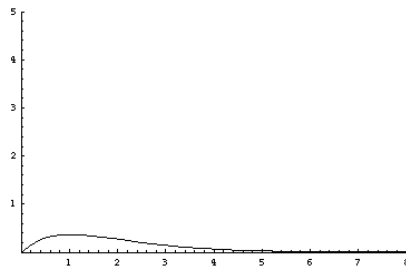
- $x=1$



$$f(x,t) = t^{x-1} e^{-t}$$

## Gamma Integrand

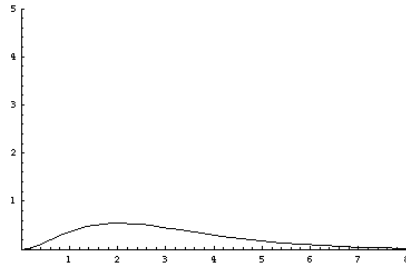
- $x=2$



$$f(x,t) = t^{x-1} e^{-t}$$

## Gamma Integrand

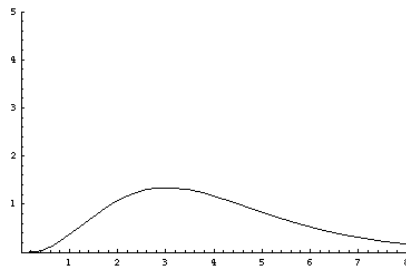
- $x=3$



$$f(x,t) = t^{x-1}e^{-t}$$

## Gamma Integrand

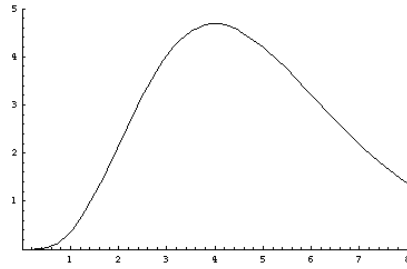
- $x=4$



$$f(x,t) = t^{x-1}e^{-t}$$

## Gamma Integrand

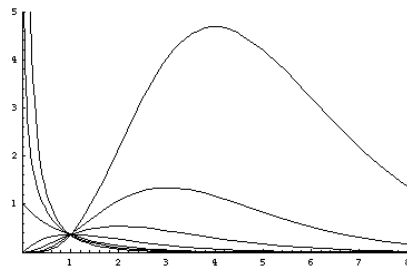
- $x=5$



$$f(x,t) = t^{x-1}e^{-t}$$

## Gamma Integrand

- $x$  is 0, 1/2, 1, 2, 3, 4, 5



$$f(x,t) = t^{x-1}e^{-t}$$

## The Gamma Function

Show  $\Gamma(x+1) = x\Gamma(x)$

$$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt \quad \text{Use integration by parts.}$$

$$u = t^x \qquad dv = e^{-t} dt$$

$$du = xt^{x-1} dt \qquad v = -e^{-t}$$

$$\Gamma(x+1) = -t^x e^{-t} \Big|_0^{\infty} - \int_0^{\infty} (-e^{-t}) xt^{x-1} dt$$

$$\Gamma(x+1) = 0 + \int_0^{\infty} xt^{x-1} e^{-t} dt = x \int_0^{\infty} t^{x-1} e^{-t} dt$$

$$\therefore \Gamma(x+1) = x\Gamma(x)$$

## The Gamma Function

How about  $\Gamma(n)$  for integer values of  $n$ ?

$$\Gamma(x+1) = x\Gamma(x) \qquad \Gamma(1) = 1$$

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1 \cdot 1 = 1!$$

$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2 \cdot 1 = 2!$$

$$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3 \cdot 2! = 3!$$

$$\Gamma(5) = \Gamma(4+1) = 4\Gamma(4) = 4 \cdot 3! = 4!$$

⋮

$$\Gamma(n+1) = \Gamma(n+1) = n\Gamma(n) = n \cdot (n-1)! = n!$$

# The Factorial Function

The result  $\Gamma(n+1) = n!$  suggests a way to extend the domain of the factorial function beyond the non-negative integers.

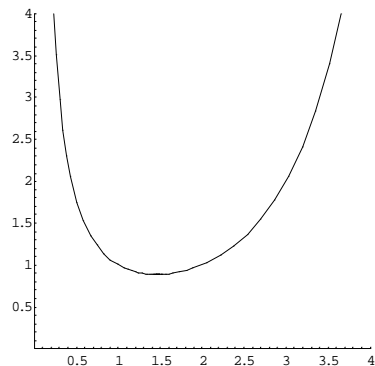
We define the factorial function for real  $x > -1$  by

$$x! = \Gamma(x+1)$$

Before finding some new values of the factorial function we need one special value of the gamma function. We need to evaluate  $\Gamma(1/2)$ .

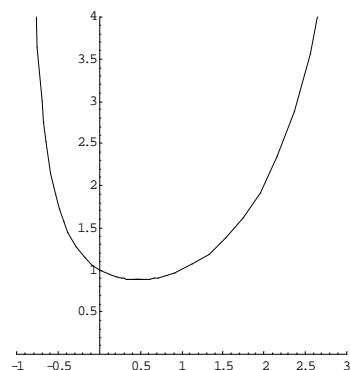
## Gamma Function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$



## Factorial Function

$$x! = \Gamma(x+1)$$



$$\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \Rightarrow$$

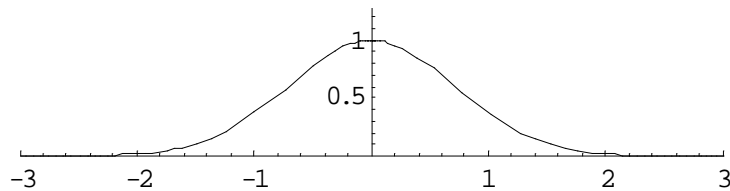
$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} t^{\frac{1}{2}-1} e^{-t} dt = \int_0^{\infty} t^{-1/2} e^{-t} dt$$

$$\text{Letting } u = t^{1/2} \Rightarrow t = u^2, \quad du = (1/2)t^{-1/2} dt$$

also  $u$  goes from 0 to  $\infty$  as  $t$  goes from 0 to  $\infty$

$$\int_0^{\infty} t^{-1/2} e^{-t} dt = \int_0^{\infty} e^{-u^2} 2 du = 2 \int_0^{\infty} e^{-u^2} du = \int_{-\infty}^{\infty} e^{-u^2} du$$

$$f(x) = e^{-x^2}$$



## Two Views

“A mathematician is one to whom  $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$  is as obvious as that twice two makes four is to you.”

Lord Kelvin

“Many things are not accessible to intuition at all, the value of  $\int_{-\infty}^{\infty} e^{-x^2} dx$  for instance.”

J. E. Littlewood

$$\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$$

We have considered two options for trying to get the students to accept this:

1. a numerical exploration
2. a proof that follows from the properties of a sequence of integrals that we investigate later.

## Factorials

We now go back to the definition  $x! = \Gamma(x+1)$

Using the recursion  $\Gamma(x+1) = x\Gamma(x)$  we can show that

$$x! = \Gamma(x+1) = x\Gamma(x) = x\Gamma((x-1)+1) = x(x-1)!$$

now with  $x! = x(x-1)!$  and  $\Gamma(1/2) = \sqrt{\pi}$  we can evaluate

$$\left(\frac{1}{2}\right)!, \left(\frac{2}{2}\right)!, \left(\frac{3}{2}\right)!, \left(\frac{4}{2}\right)!, \left(\frac{5}{2}\right)!, \left(\frac{6}{2}\right)!, \dots, \left(\frac{n}{2}\right)!$$

## Factorials

$$\left(\frac{1}{2}\right)! = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\left(\frac{2}{2}\right)! = (1)! = 1$$

$$\left(\frac{3}{2}\right)! = \left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)! = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)! = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$$

$$\left(\frac{4}{2}\right)! = \left(\frac{4}{2}\right)\left(\frac{2}{2}\right)! = \left(\frac{4}{2}\right)(1)$$

$$\left(\frac{5}{2}\right)! = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)! = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$$

$$\left(\frac{6}{2}\right)! = \left(\frac{6}{2}\right)\left(\frac{4}{2}\right)! = \left(\frac{6}{2}\right)(2)(1)$$

$$\left(\frac{7}{2}\right)! = \left(\frac{7}{2}\right)\left(\frac{5}{2}\right)! = \left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$$

$$\left(\frac{n}{2}\right)!$$

for  $n$  even  $\left(\frac{n}{2}\right)! = \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots(3)(2)(1)$   
an ordinary integer factorial

for  $n$  odd  $\left(\frac{n}{2}\right)! = \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$   
we march nicely down to  $1/2$  and end with a  $\sqrt{\pi}$

## n-Space, n-Balls and n-Spheres

Euclidean  $n$ -dimensional space is the set of ordered Real  $n$ -tuples together with a Euclidean distance measure.

$$\mathbf{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in \mathbf{R}\}$$

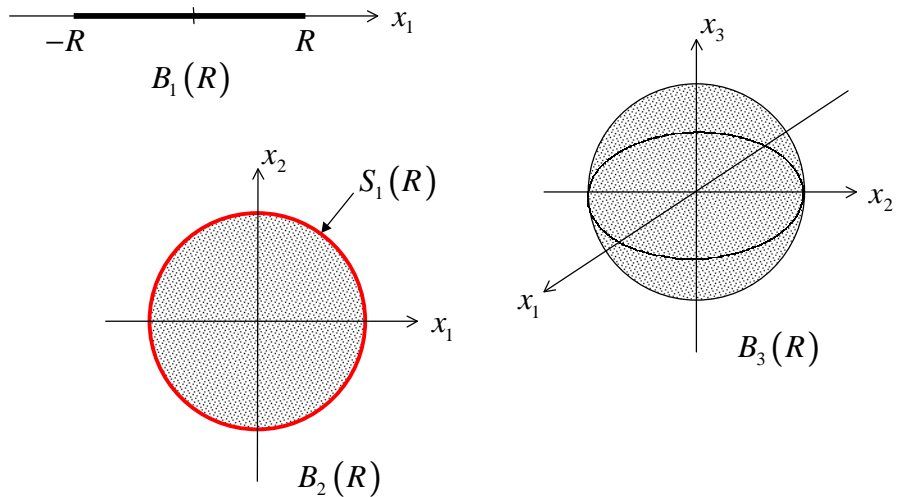
with the distance from the origin to  $(x_1, \dots, x_n) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$

We define the  $n$ -ball,  $B_n(R)$ , of radius  $R$  and its boundary,  $S_{n-1}(R)$  by

$$B_n(R) = \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2\}$$

$$S_{n-1}(R) = \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = R^2\}$$

### 1-ball, 2-ball, 3-ball, ...



### 1-ball, 2-ball, 3-ball, ...

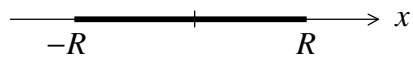
Some notation then some examples

let:  $V_n(R)$  = The " $n$ -volume" of  $B_n(R)$

$A_{n-1}(R)$  = The " $n-1$ -volume" of  $S_{n-1}(R)$

Examples

$$n=1 \quad B_1(R) = \{x \in \mathbf{R} \mid x^2 \leq R^2\} = \{x \in \mathbf{R} \mid -R \leq x \leq R\}$$

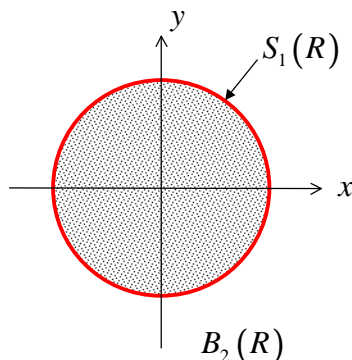


$$V_1(R) = 2R \quad A_0(R) = ?$$

## 1-ball, 2-ball, 3-ball, ...

Examples

$$n=2 \quad B_2(R) = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq R^2\}$$

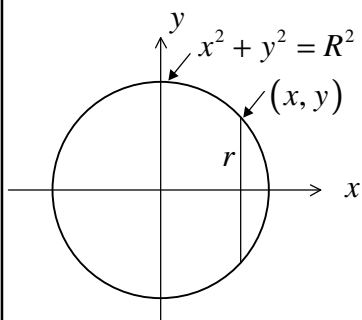


$$V_2(R) = \pi R^2$$

$$A_1(R) = 2\pi R$$

## 1-ball, 2-ball, 3-ball, ...

Computing  $V_2(R)$  using the "disk" method



$$\Delta V = 2r\Delta x$$

$$r = y = \sqrt{R^2 - x^2}$$

$$dV = 2\sqrt{R^2 - x^2} dx$$

$$V_2(R) = \int_{-R}^R dV = \int_{-R}^R 2\sqrt{R^2 - x^2} dx$$

$$= 2 \int_{-R}^R \sqrt{R^2 - x^2} dx = 4 \int_0^R \sqrt{R^2 - x^2} dx$$

using the trigonometric substitution

$$x = R \sin \theta, \quad dx = R \cos \theta d\theta$$

$$R^2 - x^2 = R^2 (1 - \sin^2 \theta) = R^2 \cos^2 \theta$$

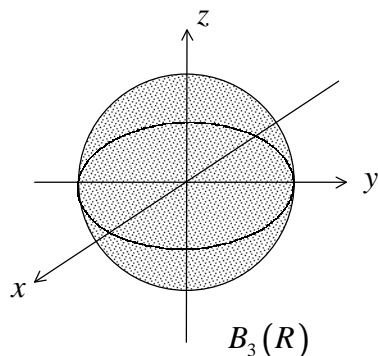
$x$  goes from 0 to  $R$  as  $\theta$  goes from 0 to  $\frac{\pi}{2}$

$$V_2(R) = 4 \int_0^{\frac{\pi}{2}} R^2 \cos^2 \theta d\theta = 4R^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

## 1-ball, 2-ball, 3-ball, ...

Examples

$$n = 3 \quad B_3(R) = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 \leq R^2\}$$

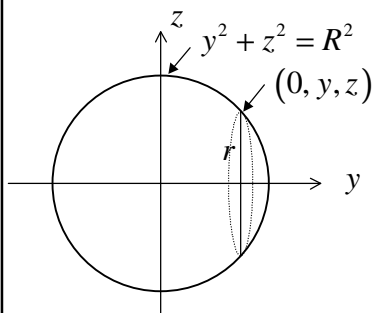


$$V_3(R) = \frac{4}{3}\pi R^3$$

$$A_2(R) = 4\pi R^2$$

## 1-ball, 2-ball, 3-ball, ...

Computing  $V_3(R)$  using the disk method



$$\Delta V = \pi r^2 \Delta y$$

$$r = z = \sqrt{R^2 - y^2}$$

$$dV = \pi (R^2 - y^2) dy$$

$$V_3(R) = \int_{-R}^R dV = \int_{-R}^R \pi (R^2 - y^2) dy$$

$$= 2\pi \int_0^R (R^2 - y^2) dy$$

using the trigonometric substitution

$$y = R \sin \theta, \quad dy = R \cos \theta d\theta$$

$$R^2 - y^2 = R^2 (1 - \sin^2 \theta) = R^2 \cos^2 \theta$$

$y$  goes from 0 to  $R$  as  $\theta$  goes from 0 to  $\frac{\pi}{2}$

$$V_3(R) = 2\pi R^3 \int_0^{\frac{\pi}{2}} \cos^3 \theta d\theta$$

recall  $V_2(R) = 4R^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$

1-ball, 2-ball, 3-ball, ...

$n$	$V_n(R)$	$A_{n-1}(R)$
0	?	?
1	$2R$	?
2	$\pi R^2$	$2\pi R$
3	$\frac{4\pi}{3} R^3$	$4\pi R^2$
4	?	?
·	·	·
·	·	·
·	·	·

1-ball, 2-ball, 3-ball, 4-ball, ...

Examples

$$n = 4 \quad B_4(R) = \{(x, y, z, w) \in \mathbf{R}^4 \mid x^2 + y^2 + z^2 + w^2 \leq R^2\}$$

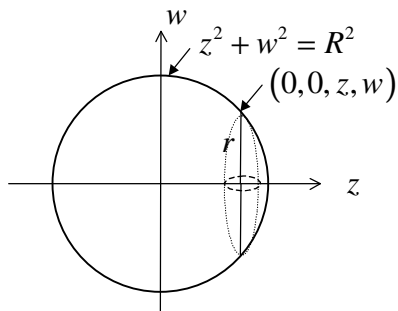
?

$$V_4(R) = ? R^4$$

$$A_3(R) = ?? R^3$$

## 1-ball, 2-ball, 3-ball, 4-ball, ...

Computing  $V_4(R)$  using the disk method



When we slice the 4-ball at a fixed value of  $z$

$$\{(x, y, z, w) \in \mathbf{R}^4 \mid x^2 + y^2 + z^2 + w^2 \leq R^2\}$$

reduces to

$$\{(x, y, z, w) \mid x^2 + y^2 + w^2 \leq R^2 - z^2 = r^2\}.$$

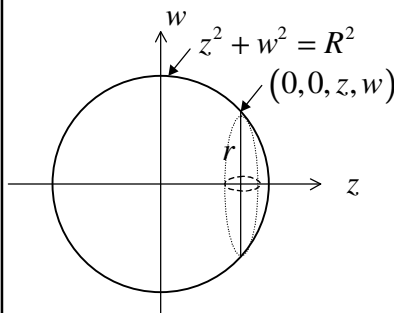
$$** \quad \Delta V = \frac{4}{3} \pi r^3 \Delta z$$

$$r = w = \sqrt{R^2 - z^2}$$

$$dV = \frac{4}{3} \pi \left( \sqrt{R^2 - z^2} \right)^3 dz$$

## 1-ball, 2-ball, 3-ball, 4-ball, ...

Computing  $V_4(R)$  using the disk method



$$** \quad \Delta V = \frac{4}{3} \pi r^3 \Delta z$$

$$r = w = \sqrt{R^2 - z^2}$$

$$dV = \frac{4}{3} \pi \left( \sqrt{R^2 - z^2} \right)^3 dz$$

$$V_4(R) = \int_{-R}^R dV = \int_{-R}^R \frac{4}{3} \pi \left( \sqrt{R^2 - z^2} \right)^3 dz$$

$$= \frac{8}{3} \pi \int_0^R \left( \sqrt{R^2 - z^2} \right)^3 dz$$

using the trigonometric substitution

$$z = R \sin \theta, \quad dz = R \cos \theta d\theta$$

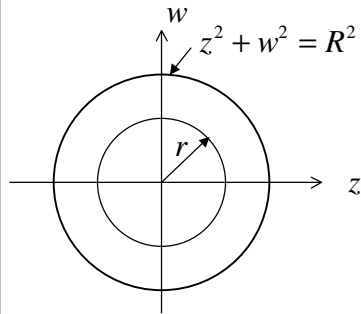
$$R^2 - z^2 = R^2 (1 - \sin^2 \theta) = R^2 \cos^2 \theta$$

$z$  goes from 0 to  $R$  as  $\theta$  goes from 0 to  $\frac{\pi}{2}$

$$V_4(R) = \frac{8}{3} \pi R^4 \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta$$

## 1-ball, 2-ball, 3-ball, 4-ball, ...

Computing  $V_4(R)$  using the shell method



$$V_4(R) = \int_0^R dV = \int_0^R A_3(r) dr$$

$$V_4(R) = \int_0^R A_3(r) dr \text{ is in the form}$$

$$F(x) = \int_0^x f(t) dt \text{ hence by the}$$

Fundamental Theorem of Calculus

$$\frac{d}{dR}(V_4) = V_4'(R) = A_3(R)$$

$$** \Delta V = A_3(r) \Delta r$$

$$dV = A_3(r) dr$$

## ... , n-ball, ...

$$B_n(R) = \{(x_1, \dots, x_{n-1}, x_n) \in \mathbf{R}^n \mid x_1^2 + \dots + x_{n-1}^2 + x_n^2 \leq R^2\}$$

?

$$V_n(R) = ? R^n$$

$$A_{n-1}(R) = ?? R^{n-1}$$

We can be more explicit

$$V_n(R) = c_n R^n$$

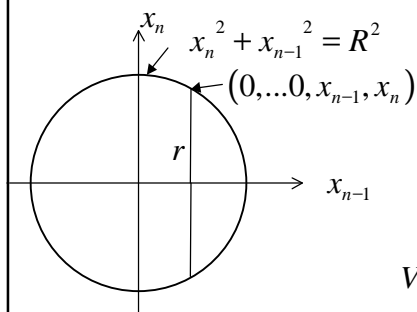
$$A_{n-1}(R) = V_n'(R) = n c_n R^{n-1}$$

1-ball, 2-ball, 3-ball, 4-ball, ... , n-ball, ...

$n$	$V_n(R)$	$A_{n-1}(R)$
0	?	?
1	$2R$	2
2	$\pi R^2$	$2\pi R$
3	$\frac{4\pi}{3} R^3$	$4\pi R^2$
4	$c_4 R^4$	$4c_4 R^3$
⋮	⋮	⋮
⋮	⋮	⋮
⋮	⋮	⋮
$n$	$c_n R^n$	$nc_n R^{n-1}$

... , n-ball, ...

again using the disk method



$$\Delta V = V_{n-1}(r) \Delta x_{n-1} = V_{n-1}(x_n) \Delta x_{n-1}$$

$$r = x_n = \sqrt{R^2 - x_{n-1}^2}$$

$$dV = V_{n-1}(\sqrt{R^2 - x_{n-1}^2}) dx_{n-1}$$

$$V_n(R) = \int_{-R}^R dV = \int_{-R}^R V_{n-1}(\sqrt{R^2 - x_{n-1}^2}) dx_{n-1}$$

$$= 2 \int_0^R V_{n-1}(\sqrt{R^2 - x_{n-1}^2}) dx_{n-1}$$

... , n-ball, ...

$$V_n(R) = \int_{-R}^R dV = 2 \int_0^R V_{n-1}(r) dx_{n-1} = 2 \int_0^R V_{n-1}(\sqrt{R^2 - x_{n-1}^2}) dx_{n-1}$$

\*\* letting  $V_n(r) = c_n r^n$ , we have  $V_{n-1}(r) = c_{n-1} r^{n-1}$  and

$$V_n(R) = 2 \int_0^R V_{n-1}(\sqrt{R^2 - x_{n-1}^2}) dx_{n-1} = 2c_{n-1} \int_0^R (\sqrt{R^2 - x_{n-1}^2})^{n-1} dx_{n-1}$$

now using the trigonometric substitution  $x_{n-1} = R \sin \theta$  implies  $dx_{n-1} = R \cos \theta d\theta$  and  $R^2 - x_{n-1}^2 = R^2(1 - \sin^2 \theta) = R^2 \cos^2 \theta$ .

$$\text{So } V_n(R) = 2c_{n-1} R^n \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta = 2c_{n-1} I_n R^n, \quad \text{where } I_n = \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta$$

Hence  $V_n(R) = c_n R^n = 2c_{n-1} I_n R^n$ , which implies

$$c_n = 2c_{n-1} I_n$$

### Some Interesting and useful Integrals

$$\text{Let } I_n = \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta$$

$$\text{Show } I_0 = \frac{\pi}{2}, \quad I_1 = 1$$

$$\text{Show } \int \cos^n \theta d\theta = \frac{1}{n} \sin \theta \cos^{n-1} \theta + \frac{n-1}{n} \int \cos^{n-2} \theta d\theta$$

$$\text{Show } \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} \theta d\theta$$

$$\text{Hence } I_n = \frac{n-1}{n} I_{n-2}$$

## $I_n$ for even and odd $n$

$$I_0 = \frac{\pi}{2}, \quad I_1 = 1 \quad \text{and} \quad I_n = \frac{n-1}{n} I_{n-2}$$

$$I_0 = \frac{\pi}{2}$$

$$I_1 = 1$$

$$I_2 = \frac{1}{2} I_0 = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I_3 = \frac{2}{3} I_1 = \frac{2}{3}$$

$$I_4 = \frac{3}{4} I_2 = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I_5 = \frac{4}{5} I_3 = \frac{4}{5} \cdot \frac{2}{3}$$

$$I_6 = \frac{5}{6} I_4 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I_7 = \frac{6}{7} I_5 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

⋮

⋮

⋮

⋮

⋮

⋮

$$I_{2k} = \frac{2k-1}{2k} \cdot \frac{2k-3}{2(k-1)} \cdot \frac{2k-5}{2(k-2)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I_{2k+1} = \frac{2k}{2k+1} \cdot \frac{2(k-1)}{2k-1} \cdot \frac{2(k-2)}{2k-3} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

## Back to the n-ball

$$V_n(R) = c_n R^n, \quad c_n = 2c_{n-1} I_n \quad \text{also} \quad V_1(R) = 2R \Rightarrow c_1 = 2$$

$$c_2 = 2c_1 I_2 = 2(2) \left( \frac{\pi}{4} \right) = \pi$$

$$c_8 = \frac{1}{4 \cdot 3 \cdot 2} \pi^4$$

$$c_3 = 2c_2 I_3 = 2(\pi) \left( \frac{2}{3} \right) = \frac{4}{3} \pi$$

$$c_9 = \frac{2^5}{9 \cdot 7 \cdot 5 \cdot 3} \pi^4$$

$$c_4 = 2c_3 I_4 = 2 \left( \frac{4}{3} \pi \right) \left( \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right) = \frac{1}{2} \pi^2$$

$$c_{10} = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} \pi^5$$

$$c_5 = 2c_4 I_5 = 2 \left( \frac{1}{2} \pi^2 \right) \left( \frac{4}{5} \cdot \frac{2}{3} \right) = \frac{8}{5 \cdot 3} \pi^2$$

⋮

$$c_6 = 2c_5 I_6 = 2 \left( \frac{4}{5} \cdot \frac{2}{3} \pi^2 \right) \left( \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right) = \frac{1}{3 \cdot 2} \pi^3$$

$$c_n = \begin{cases} \frac{\pi^{n/2}}{(n/2)!} & \text{if } n \text{ is even} \\ \frac{??}{???} & \text{if } n \text{ is odd} \end{cases}$$

$$c_7 = 2c_6 I_7 = 2 \left( \frac{1}{6} \pi^3 \right) \left( \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \right) = \frac{16}{7 \cdot 5 \cdot 3} \pi^3$$

## Back to the n-ball

$$c_1 = 2$$

$$c_3 = \frac{4}{3}\pi$$

$$c_5 = \frac{8}{5 \cdot 3}\pi^2$$

$$c_7 = \frac{2^4}{7 \cdot 5 \cdot 3}\pi^3$$

$$c_9 = \frac{2^5}{9 \cdot 7 \cdot 5 \cdot 3}\pi^4 = \frac{\pi^4 \sqrt{\pi}}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}} \sqrt{\pi} = \frac{\pi^{9/2}}{(9/2)!}$$

⋮

$$c_n = \frac{\pi^{n/2}}{(n/2)!} \text{ if } n \text{ is odd so } c_n = \frac{\pi^{n/2}}{(n/2)!} \text{ if } n \text{ is even or odd}$$

$$c_n = \frac{\pi^{n/2}}{(n/2)!} \text{ if } n \text{ is even}$$

What about  $n$  odd ?

## Finally

$$V_n(R) = c_n R^n = \frac{\pi^{n/2}}{(n/2)!} R^n$$

$$A_{n-1}(R) = n \frac{\pi^{n/2}}{(n/2)!} R^{n-1}$$

$n$	$V_n(R)$	$A_{n-1}(R)$
0	1	
1	$2R$	2
2	$\pi R^2$	$2\pi R$
3	$\frac{4\pi}{3} R^3$	$4\pi R^2$
4	$\frac{\pi^2}{2} R^4$	$2\pi^2 R^3$
5	$\frac{8\pi^2}{5} R^5$	$8\pi^2 R^4$
6	$\frac{\pi^3}{6} R^6$	$\pi^3 R^5$
⋮	⋮	⋮
⋮	⋮	⋮
⋮	⋮	⋮

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