

Trigonometry, Geometry, Algebra and Complex Numbers

Dedicated to David Cohen (1942 - 2002)

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“The shortest path between two truths in the real domain passes through the complex domain.”

-*Jacques Hadamard* (1865-1963) (as quoted in Stillwell)

“The advanced reader who skips parts that appear too elementary may miss more than the less advanced reader who skips parts that appear too complex.”

-*G Polya* (as quoted in Graham Knuth and Patashnik)

Which Truths in the Real Domain?

$$\sin\left(\frac{\pi}{2}\right) = 1 = \frac{2}{2}$$

$$\sin\left(\frac{\pi}{3}\right) \cdot \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3}{4}$$

$$\sin\left(\frac{\pi}{4}\right) \cdot \sin\left(\frac{2\pi}{4}\right) \cdot \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} \cdot 1 \cdot \frac{\sqrt{2}}{2} = \frac{2}{4} = \frac{4}{8}$$

$$\sin\left(\frac{\pi}{5}\right) \cdot \sin\left(\frac{2\pi}{5}\right) \cdot \sin\left(\frac{3\pi}{5}\right) \cdot \sin\left(\frac{4\pi}{5}\right) = \alpha \cdot \beta \cdot \beta \cdot \alpha = \frac{5}{16}$$

...

$$\sin\left(\frac{\pi}{n}\right) \cdot \sin\left(\frac{2\pi}{n}\right) \cdot \dots \cdot \sin\left(\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}$$

Introduction

- What are our goals?

The primary goal is to look at a beautiful piece of mathematics. We also hope to shed some light on connections between different areas of mathematics.

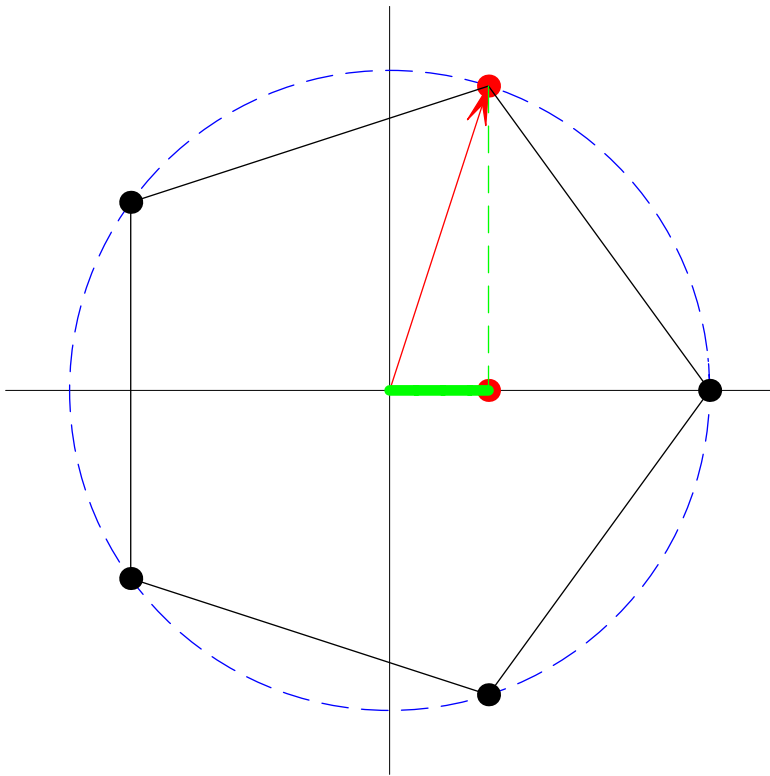
- Where are we going?

A proof of the identity on the previous slide using trigonometry, geometry, algebra, and complex numbers.

- What will we see along the way?

A proof that a regular pentagon can be constructed with a compass and straight edge using trigonometry, geometry, algebra, and complex numbers.

Constructing a Regular Pentagon



$$\cos\left(\frac{2\pi}{5}\right) = ???$$

Fundamental Theorem of Algebra

- Fundamental Theorem of Arithmetic: any integer r can be uniquely factored into (not necessarily distinct) primes:

$$r = p_1 \cdot p_2 \cdot \dots \cdot p_n$$

- Fundamental Theorem of Algebra: over the complex numbers any n -th degree polynomial $p(z)$ can be uniquely factored into n (not necessarily distinct) linear factors:

$$p(z) = \alpha(z - z_1)(z - z_2)\dots(z - z_n)$$

Fundamental Theorem of Algebra

- Fundamental Theorem of Algebra: over the complex numbers any n -th degree polynomial $p(z)$ can be uniquely factored into n (not necessarily distinct) linear factors:

$$p(z) = \alpha(z - z_1)(z - z_2)\dots(z - z_n)$$

- Note $p(z)$ can be thought of as an algebraic object or as a function whose domain is the set of complex numbers.
- If we do consider a function, $f(z) = p(z)$, then z_1, z_2, \dots, z_n form the set of n (again not necessarily distinct) roots of the equation $p(z) = 0$ (or *zeros* of $p(z)$).

Factoring and the Fundamental Theorem of Algebra – Examples

$$f_1(z) = z^2 - 1 = (z - 1)(z + 1)$$

\implies zeros at 1 and -1

$$f_2(z) = z^2 + 1 = z^2 - (-1) = (z - i)(z + i)$$

\implies zeros at i and $-i$

Define i as a number whose square is -1 .

$$f_3(z) = z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$$

\implies zeros at 1 and let's dig some more

Complex Numbers

$$z = a + ib \quad w = c + id$$

- Equality: $z = w \iff a = c \text{ and } b = d$
- Addition: $z + w = (a + c) + i(b + d)$
- Multiplication:
 $z \cdot w = ac + iad + ibc + i^2bd$
 $z \cdot w = ac + i(ad + bc) - bd$
 $z \cdot w = (ac - bd) + i(ad + bc)$
- Conjugate: $\bar{z} = a - ib$
- Absolute Value (or length): $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$

Geometry of Complex Arithmetic

- GSP Examples
- Addition of complex numbers vector addition
- Multiplication by a real number: $t(a + ib) = ta + itb$
- dilation (stretching):
$$|tz| = \sqrt{((ta)^2 + (tb)^2)} = |t|(\sqrt{a^2 + b^2}) = |t||z|$$

Geometry of Complex Multiplication

- Multiplication by a real number:

$$|tz| = |t||z|$$

- Multiplication by i as a 90° rotation
- Complex Multiplication: Adding angles - multiplying lengths

Multiplying Polar Representations

$$z = r(\cos \theta + i \sin \theta) \quad w = s(\cos \phi + i \sin \phi)$$

$$\text{Recall } z \cdot w = (ac - bd) + i(ad + bc)$$

$$\begin{aligned} zw &= rs((\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\cos \theta \sin \phi + \sin \theta \cos \phi)) \\ &= rs((\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\sin \theta \cos \phi + \cos \theta \sin \phi)) \end{aligned}$$

The cheer: “sin cos cos sin cos cos sin sin!”

$$\therefore zw = rs(\cos(\theta + \phi) + i \sin(\theta + \phi))$$

Lengths

$$z = r(\cos \theta + i \sin \theta) \quad w = s(\cos \phi + i \sin \phi)$$

$$(\text{note } r = |z|) \quad (s = |w|)$$

$$zw = rs(\cos(\theta + \phi) + i \sin(\theta + \phi))$$

$$|zw| = rs = |z||w|$$

Angles and DeMoivre's Theorem

$$z = r(\cos \theta + i \sin \theta) \quad w = s(\cos \phi + i \sin \phi)$$

$$zw = rs(\cos(\theta + \phi) + i \sin(\theta + \phi))$$

Powers of $z = r(\cos \theta + i \sin \theta)$:

$$z^2 = r^2(\cos 2\theta + i \sin 2\theta)$$

$$z^3 = r^3(\cos 3\theta + i \sin 3\theta)$$

...

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

DeMoivre's Theorem: $[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$

Using DeMoivre's Theorem: Find Fifth Roots of Unity

$$z^5 = 1$$

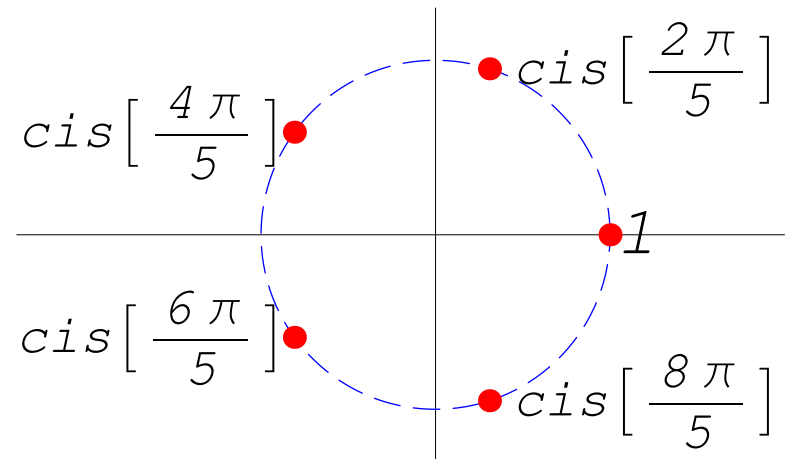
$$(\cos \theta + i \sin \theta)^5 = 1$$

$$(\cos 5\theta + i \sin 5\theta) = 1 = 1 + i0$$

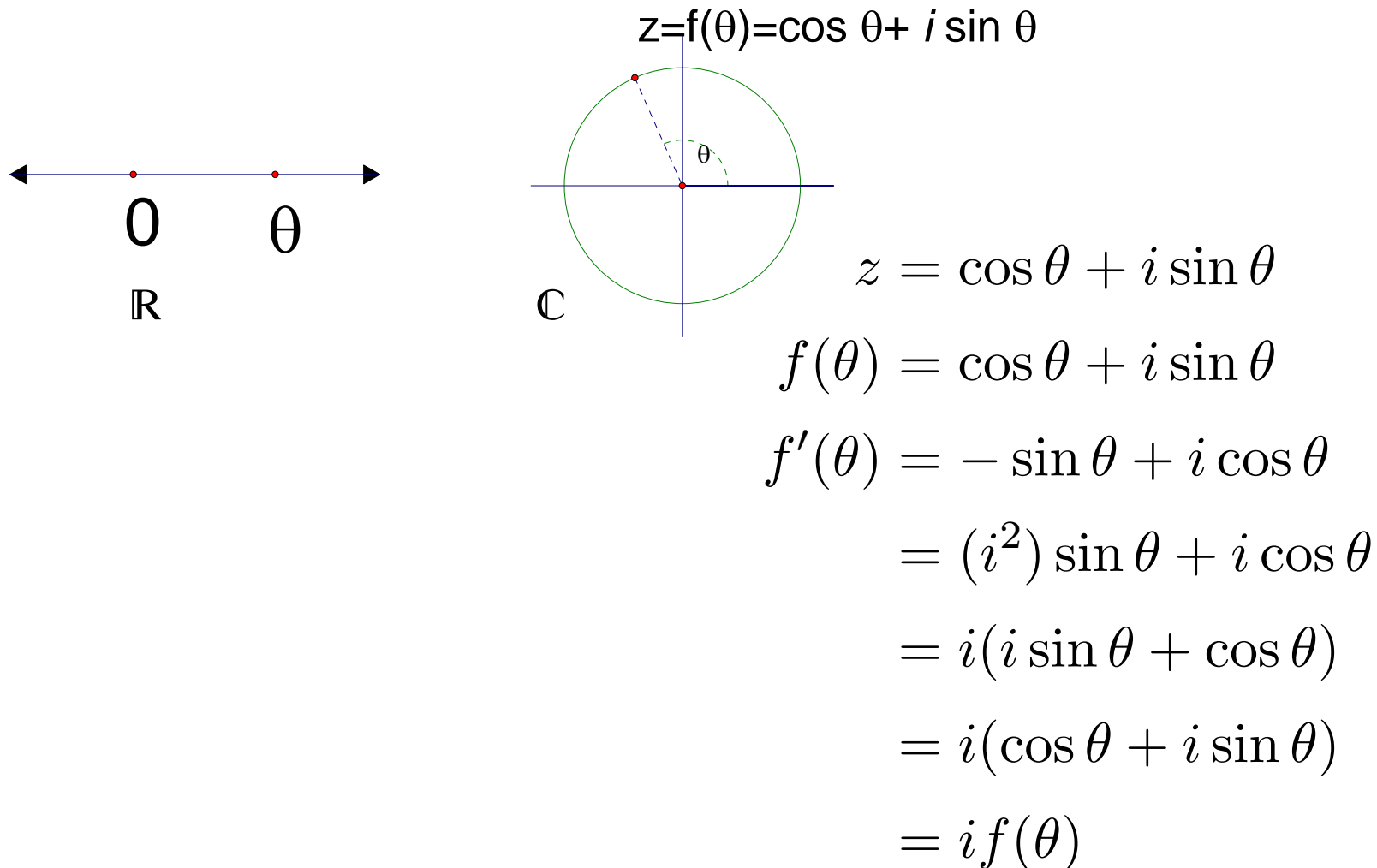
$$\cos 5\theta = 1 \text{ and } \sin 5\theta = 0$$

$$5\theta = k \cdot 2\pi$$

$$\therefore \theta = \frac{k \cdot 2\pi}{5}$$



Parametric Unit Circle



Rewriting $f(\theta)$

$$f(\theta) = \cos \theta + i \sin \theta \quad f'(\theta) = i f(\theta)$$

$$f(0) = \cos 0 + i \sin 0 = 1$$

So f satisfies a differential equation of the form:

$$f'(\theta) = k f(\theta) \implies f(\theta) = D e^{k\theta}$$

with initial condition: $f(0) = 1 \implies 1 = D e^{k0}$

$$\text{So } D = 1 \text{ and } f'(\theta) = i f(\theta) \implies k = i$$

By analogy with real valued functions:

$$f(\theta) = e^{i\theta}$$

Another Polar Notation

Our function $f(\theta) = e^{i\theta}$ leads to Euler's:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

This provides a more compact polar form, rather than

$$z = r(\cos \theta + i \sin \theta)$$

we can write

$$z = re^{i\theta}$$

Roots of $z^5 - 1 = 0$

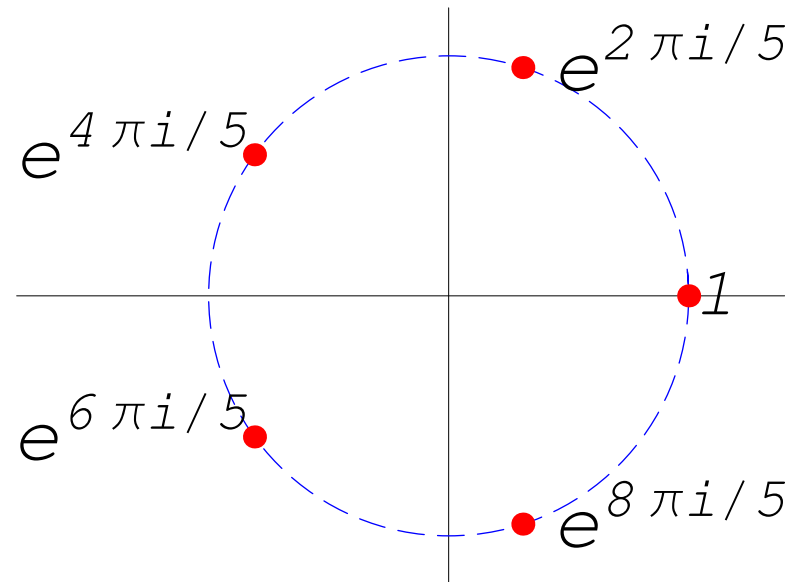
$$z^5 = 1$$

$$(e^{i\theta})^5 = 1$$

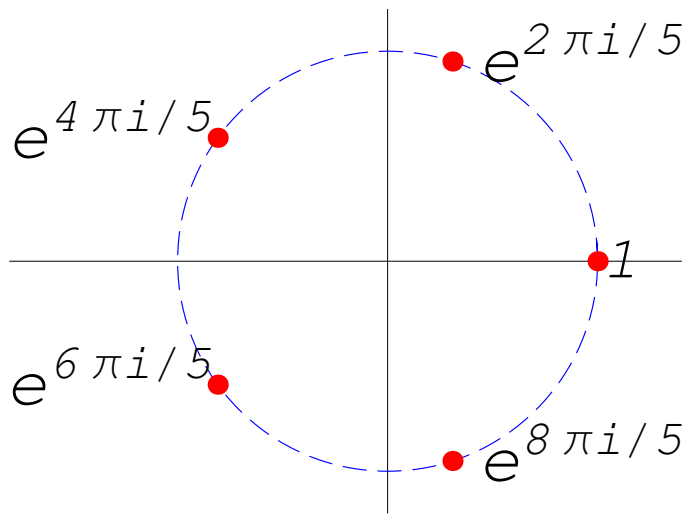
$$e^{i5\theta} = e^{ik2\pi}$$

$$i5\theta = ik \cdot 2\pi$$

$$\therefore \theta = \frac{k \cdot 2\pi}{5}$$



Factoring $z^5 - 1$



$$z^5 - 1 = (z - 1)(z - e^{\frac{2\pi i}{5}})(z - e^{\frac{4\pi i}{5}})(z - e^{\frac{6\pi i}{5}})(z - e^{\frac{8\pi i}{5}})$$

$$z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$$

So $(z^4 + z^3 + z^2 + z + 1) =$

$$(z - e^{\frac{2\pi i}{5}})(z - e^{\frac{4\pi i}{5}})(z - e^{\frac{6\pi i}{5}})(z - e^{\frac{8\pi i}{5}})$$

Conjugates on Unit Circle

$$z = \cos \theta + i \sin \theta$$

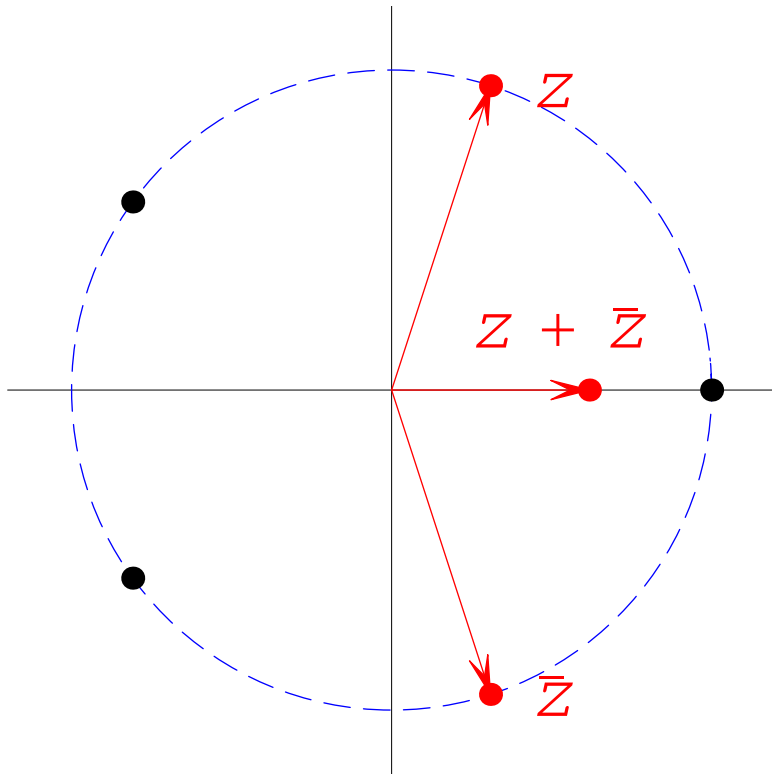
$$\bar{z} = \cos \theta - i \sin \theta$$

$$\frac{1}{z} = z^{-1}$$

$$= \cos(-\theta) + i \sin(-\theta)$$

$$= \cos(\theta) - i \sin(\theta)$$

$$z + \frac{1}{z} = z + \bar{z} = 2 \cos(\theta)$$



Given: $z^4 + z^3 + z^2 + z + 1 = 0$

Find: $\cos\left(\frac{2\pi}{5}\right)$

$$z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = (z^2 + \frac{1}{z^2} + 1) + z + \frac{1}{z} = 0 \quad (1)$$

$$\left(z + \frac{1}{z}\right)^2 = z^2 + 2 + \frac{1}{z^2} = (z^2 + \frac{1}{z^2} + 1) + 1 \quad (2)$$

$$\left(z + \frac{1}{z}\right)^2 - 1 + \left(z + \frac{1}{z}\right) = 0 \quad (3)$$

$$\left(z + \frac{1}{z}\right)^2 + \left(z + \frac{1}{z}\right) - 1 = 0 \quad (4)$$

Finding $\cos\left(\frac{2\pi}{5}\right)$

We have $\left(z + \frac{1}{z}\right)^2 + \left(z + \frac{1}{z}\right) - 1 = 0$, a quadratic

$$\left(z + \frac{1}{z}\right) = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

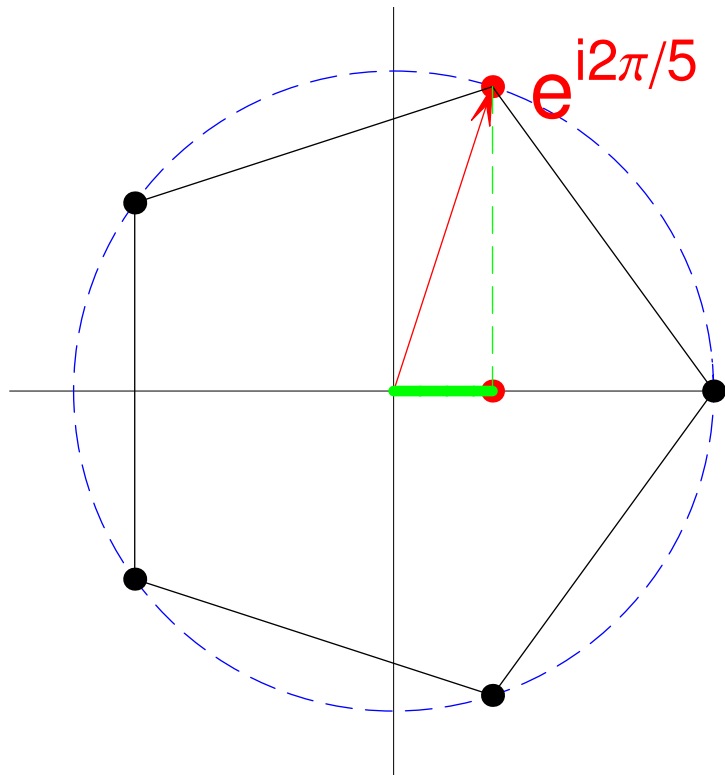
For any z of length 1, $z + \frac{1}{z} = 2\cos(\theta)$ (conjugates)

Since $\cos\left(\frac{2\pi}{5}\right)$ is positive, we have:

$$\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}$$

Using $\cos\left(\frac{2\pi}{5}\right)$

$$\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5} - 1}{4}$$



Start with a unit circle. Construct this length. (How do you get $\sqrt{5}$?) Use it to mark off the green segment on the x -axis. Construct the green dashed perpendicular to get $e^{i2\pi/5}$, and set the compass to the length of the chord from 1 to $e^{i2\pi/5}$. to mark off the other three 5^{th} roots.

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