Ellipse Tangents Problem

Problem: Given a curve specified by the equation

$$x^2 + 4y^2 = 36 \tag{1}$$

Find equation(s) for the line(s) tangent to the curve that pass through the point $(12, 3)$.

Approach: The equation gives an ellipse. Draw a sketch and you'll see that there should be two tangents that run through $(12, 3)$. We'll assume a tangent through $(12, 3)$ is tangent to the ellipse at (a, b). We can use implicit differentiation to get an expression for the slope of the tangent at (a, b). Using slope-point form, we can get an equation for the tangent in terms of a and b. Finally, we use that equation and the equation for the ellipse to solve for a and b.

Solution: Let (a, b) be a point on the graph of $x^2 + 4y^2 = 36$ which has a tangent going through $(12, 3)$. We should note that a and b have the this important relationship:

$$a^2 + 4b^2 = 36 \tag{2}$$

Implicit differentiation of $x^2 + 4y^2 = 36$ yields [1]

$$\frac{dy}{dx} = \frac{-x}{4y} \tag{2}$$

so the slope of the tangent is $\frac{-a}{4b}$. Using *point-slope form*

$$y - 3 = \frac{-a}{4b}(x - 12) \tag{3}$$

is an equation for the desired tangent line. Now we need to find values for a and b. Since (a, b) is also on the tangent line we can substitute a and b for x and y in equation (3).

$$b - 3 = \frac{-a}{4b}(a - 12)$$

[1] Implicit differentiation says just take the derivatives of both sides of the equation and solve for y':

$$D_x[x^2 + 4y^2] = D_x[36]$$
$$2x + 8y\frac{dy}{dx} = 0$$

Keeping in mind equation (2), a little algebra gets us:

$$
\begin{aligned}
(4b)(b-3) &= -a(a-12) \\
4b^2 - 12b &= -a^2 + 12a \\
4b^2 + a^2 &= 12a + 12b \\
a^2 + 4b^2 &= 12(a+b) \qquad (4)
\end{aligned}
$$

Notice that the left hand sides of equations (2) and (4) are identical. This means the right hand sides must be equal.

$$
\begin{aligned}
12(a+b) &= 36 \\
a+b &= 3 \\
b &= 3-a \qquad (5)
\end{aligned}
$$

Substituting $3-a$ for b in equation (2):

$$
\begin{aligned}
a^2 + 4(3-a)^2 &= 36 \\
a^2 + 4(9 - 6a + a^2) &= 36 \\
36 - 24a + 5a^2 &= 36 \\
5a^2 - 24a &= 0 \\
a(5a-24) &= 0
\end{aligned}
$$

This gives solutions of $a = 0$ and $a = \frac{24}{5}$.

Substituting back into equation (5) we get the points $(0,3)$ and $(\frac{24}{5}, \frac{-9}{5})$. Finally we use *Taylor form* to get the two equations for the tangent lines.

$$
\begin{aligned}
y &= 3 \qquad (6) \\
y &= 3 - \frac{2}{3}(x-12) \qquad (7)
\end{aligned}
$$

Conclusion: The equations for the two tangent lines are given by equations (6) and (7). The problem calls for some insightful algebra. Without the insight of looking for the expression $a^2 + 4b^2$ (see equations (2) and (4)) it would be easy to get lost in a deluge of algebra.

bic 10/28/03 v1